From Numerical To Analytical Amplitudes

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1.1 Motivation (1/2)

Cross sections at hadron colliders:

$$\sigma_{2\to n-2} = \sum_{a,b} \int dx_a dx_b \;\; f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F)\; \hat{\sigma}_{ab\to n-2}(\mu_F, \mu_R)$$
$$d\hat{\sigma}_n = \frac{1}{2\hat{s}} d\Pi_{n-2}\; (2\pi)^4 \delta^4 \left(\sum_{i=1}^{n} p_i \right) \overline{|\mathcal{A}(p_i, \mu_F, \mu_R)|^2}$$

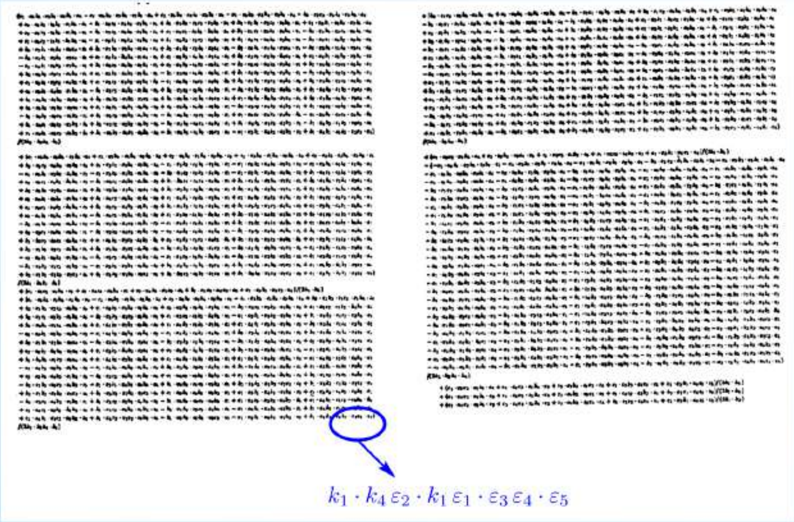
Improving the prediction requires both more loops and higher multiplicity.

The table shows the powers of the coupling:

loop\mult	**4**	**5**	**6**	**7**
0	2	3	4	5
1	4	5	6	7
2	6	7	8	9

1.1 Motivation (2/2)

Brute force calculations are a mess:



Often results are much easier:

$$A^{tree}(1_g^+ 2_g^+ 3_g^+ 4_g^- 5_g^-) = \frac{i\,\langle 45\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

1.2 Color Ordered Amplitudes (1/1)

Relation to the full amplitude @ tree level:

$$\mathcal{A}_n^{tree}(p_i, \lambda_i, a_i) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_n^{tree}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n})).$$

Color decomposition at one loop:

$$\mathcal{A}_n^{1-loop}(p_i, \lambda_i, a_i) = g^n \sum_{\sigma \in S_n/Z_n} N_c \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A_{n;1}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n}))$$
$$+ \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/Z_{n;c}} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(c-1)}}) \mathrm{Tr}(T^{a_{\sigma(c)}} \ldots T^{a_{\sigma(n)}}) A_{n;c}(\sigma(1^{\lambda_1}), \ldots, \sigma(n^{\lambda_n}))$$

Decomposition in terms of basis integrals:

$$A_{n;1}^{1-loop} = \sum_i d_i I^i_{Box} + \sum_i c_i I^i_{Triangle} + \sum_i b_i I^i_{Bubble} + R$$

1.3 Spinor Helicity (1/3)

The lowest-laying representations of the Lorentz group are:

(j_-, j_+)	**dimension**	**name**	**quantum field**	**kinematic variable**
(0, 0)	1	scalar	h	m
(0, 1/2)	2	right-handed Weyl spinor	$\chi_{R\,\alpha}$	λ_α
(1/2, 0)	2	left-handed Weyl spinor	$\chi_L^{\dot{\alpha}}$	$\bar{\lambda}^{\dot{\alpha}}$
(1/2, 1/2)	4	rank-two spinor/four vector	$A^\mu / A^{\dot{\alpha}\alpha}$	$P^\mu / P^{\dot{\alpha}\alpha}$
(1/2, 0) ⊕ (0, 1/2)	4	bispinor (Dirac spinor)	Ψ	u, v

1.3 Spinor Helicity(2/3)

Weyl spinors are sufficient to represent the kinematics of massless particles, recall:

$$\det(P^{\dot{\alpha}\alpha}) = m^2 \to 0 \quad \Longrightarrow \quad P^{\dot{\alpha}\alpha} = \bar{\lambda}^{\dot{\alpha}}\lambda^{\alpha},$$

$$\lambda_\alpha = \begin{pmatrix} \sqrt{p^0 + p^3} \\ \frac{p^1 + ip^2}{\sqrt{p^0+p^3}} \end{pmatrix}, \quad \lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \quad \bar{\lambda}^{\dot{\alpha}} = (\lambda^\alpha)^\dagger \quad \text{(for real momenta)}$$

Some definitions:

$$\langle ij\rangle = \lambda_i\lambda_j = (\lambda_i)^\alpha(\lambda_j)_\alpha \qquad [ij] = \bar{\lambda}_i\bar{\lambda}_j = (\bar{\lambda}_i)_{\dot{\alpha}}(\bar{\lambda}_j)^{\dot{\alpha}}$$

$$s_{ij} = \langle ij\rangle[ji]$$

$$\langle i \mid (j+k) \mid l] = (\lambda_i)^\alpha(\not{P}_j + \not{P}_k)_{\alpha\dot{\alpha}}\bar{\lambda}_l^{\dot{\alpha}}$$

$$\langle i \mid (j+k) \mid (l+m) \mid n\rangle = (\lambda_i)^\alpha(\not{P}_j + \not{P}_k)_{\alpha\dot{\alpha}}(\bar{\not{P}}_l + \bar{\not{P}}_m)^{\dot{\alpha}\alpha}(\lambda_n)_\alpha$$

$$tr_5(ijkl) = tr(\gamma^5\not{P}_i\not{P}_j\not{P}_k\not{P}_l) = [i\,|\,j\,|\,k\,|\,l\,|\,i\rangle - \langle i\,|\,j\,|\,k\,|\,l\,|\,i]$$

1.3 Spinor Helicity(3/3)

Examples in python:

```
oInvariants = Invariants(6)
pprint(oInvariants.invs_3[:4])
pprint(oInvariants.invs_s[:8])
```

```
[⟨1|(2+3)|1], ⟨1|(2+6)|1], ⟨1|(3+4)|1], ⟨1|(4+5)|1]]
[s_123, s_124, s_125, s_134, s_135, s_145, s_234, s_235]
```

```
oParticles = Particles(6); oParticles.fix_mom_cons(real_momenta=False)
pprint(gmpTools.to_complex(oParticles.compute("⟨1|2⟩") *
                           oParticles.compute("[2|1]")))
pprint(gmpTools.to_complex(oParticles.compute("s_12")))
```

```
(-3.29143406906+22.2901526083j)
(-3.29143406906+22.2901526083j)
```

2.1 Singular limits (1/4)

Singular limits give us information about the poles of the amplitude:

$$\langle ij \rangle \to \varepsilon, \quad f \to \varepsilon^{\alpha} \;\Rightarrow\; log(f) \to \alpha \cdot log(\varepsilon)$$

$\Rightarrow$ The slope of $log(f)(\varepsilon)$ gives us the type of singularity, if any exists.

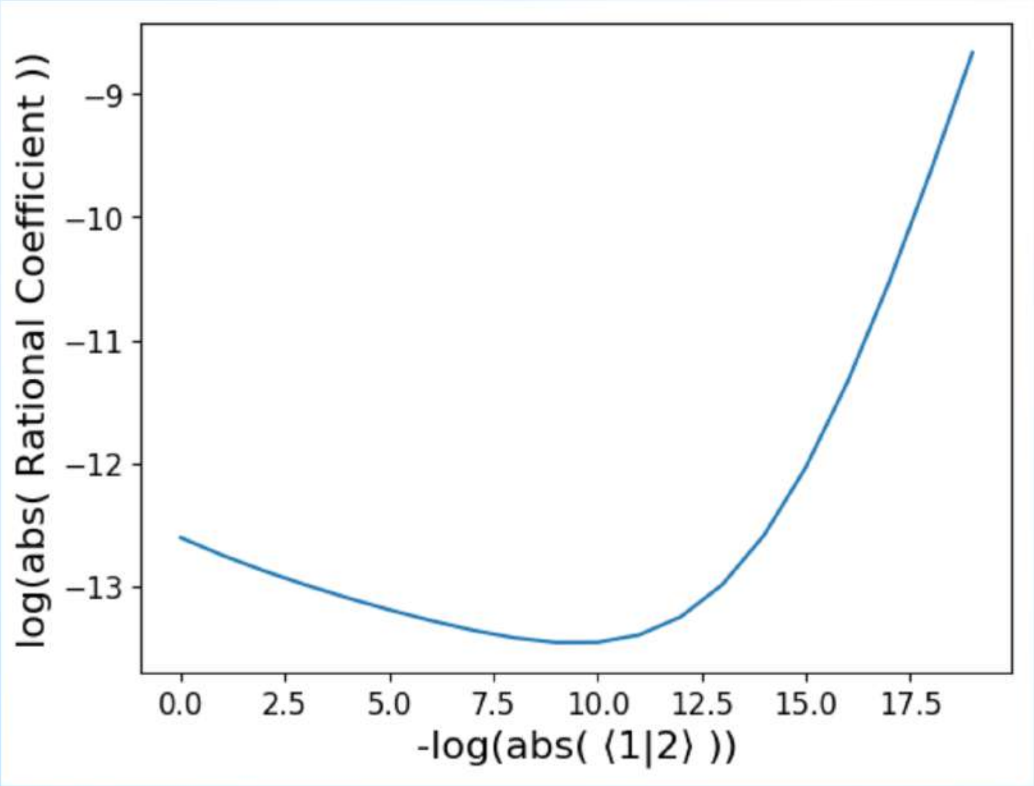
Constructing the phase space ("..." in the output below hide all O(~1) spinor variables):

```
oParticles.randomise_all(); oParticles.set("⟨1|2⟩", 10 ** -30)
oParticles.phasespace_consistency_check(oInvariants.full, silent=Fals
```

```
Consistency check:
The largest momentum violation is 2.24238986954e-307
The largest on shell violation is 6.0355128412e-307
⟨1|2⟩ = 1e-30
...
```

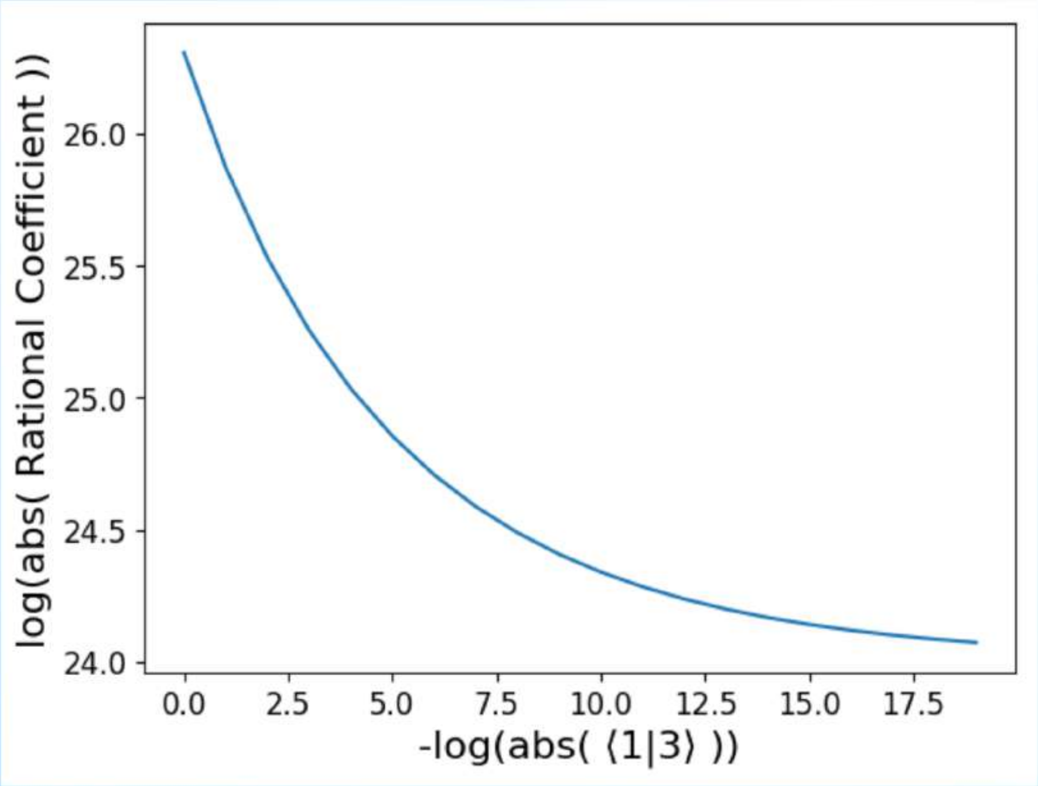
2.1 Singular limits (2/4)

```
# Simple pole: plot A(pmpmpm) in lim ⟨1|2⟩ → 0 ↔
```



2.1 Singular limits (3/4)

```
# Not a pole: plot A(pmpmpm) in lim ⟨1|3⟩ → 0↔
```



2.1 Singular limits (4/4)

Computing this slope for all invariants gives us the full list of poles and their order:

```
oUnknown.do_single_collinear_limits();
```

```
The least common denominator is
/⟨1|2⟩[1|2]⟨1|6⟩[1|6]⟨2|3⟩[2|3]⟨3|4⟩[3|4]⟨4|5⟩[4|5]⟨5|6⟩
[5|6]s_123s_234s_345

Mass dimension & phase weights: -2, [-2, 2, -2, 2, -2, 2]
→  16, [-2, 2, -2, 2, -2, 2]
```

The complexity of the numerator ansatz depends on the mass dimension.
A mass dimension of ~16 implies an ansatz of $\mathcal{O}(10^4)$ terms, which is not ideal.
Smaller denominators (i.e. a clearer pole structure) would imply easier numerators.

2.2 Doubly singular limits (1/4)

Constructing doubly singular limits is similar, but the phase space will be less "clean":

```
oParticles.randomise_all()
oParticles.set_pair("⟨1|2⟩", 10 ** -30, "⟨2|3⟩", 10 ** -30)
oParticles.phasespace_consistency_check(oInvariants.full, silent=False
```

```
Consistency check:
The largest momentum violation is 2.22507385851e-308
The largest on shell violation is 1.11253692925e-307
⟨3|(1+2)|4] = 1.5432242072e-31
⟨3|(1+2)|(2+4)|3⟩ = 1.62794486639e-31
⟨1|(2+3)|(2+6)|1⟩ = 3.80546997131e-31
⟨1|(2+3)|(3+5)|4⟩ = 4.75196693403e-31
⟨1|3⟩ = 6.23276815214e-31
⟨1|(2+3)|6] = 8.06317867575e-31
⟨1|(2+3)|(3+4)|5⟩ = 8.30471602635e-31
⟨1|2⟩ = 1e-30
⟨2|3⟩ = 1e-30
⟨2|(1+3)|(1+6)|2⟩ = 1.17643508678e-30
⟨3|(1+2)|5] = 1.20019665595e-30
⟨2|(1+3)|6] = 1.58676359058e-30
```

2.2 Doubly singular limits (2/4)

Reconstructing the behaviour in the limit involves again the slope of a log plot.

```
# Showing: scaling in limit / degeneracy of phase space / cleaned ps
oUnknown.do_double_collinear_limits(silent=True)
oUnknown.collinear_data
```

	⟨1|2⟩	[1|2]	⟨1|6⟩	[1|6]	⟨2|3⟩	[2|3]	⟨3|4⟩	[
⟨1|2⟩	1	1/2/2	1/30/5	1/3/2	1/31/5	1/3/2	1/2/2	2/
[1|2]	1/2/2	1	1/3/2	1/31/5	1/3/2	1/30/5	2/12/3	1/
⟨1|6⟩	1/30/5	1/3/2	1	1/2/2	1/2/2	2/12/3	1/10/2	2/
[1|6]	1/3/2	1/31/5	1/2/2	1	2/12/3	1/2/2	2/4/2	1/
⟨2|3⟩	1/31/5	1/3/2	1/2/2	2/12/3	1	1/2/2	1/30/6	1/
[2|3]	1/3/2	1/30/5	2/12/3	1/2/2	1/2/2	1	1/3/2	1/
⟨3|4⟩	1/2/2	2/12/3	1/10/2	2/4/2	1/30/6	1/3/2	1	1/
[3|4]	2/12/3	1/2/2	2/4/2	1/10/2	1/3/2	1/31/5	1/2/2	1
⟨4|5⟩	1/10/2	2/3/2	1/2/2	2/12/3	1/2/2	2/12/3	1/31/5	1/

2.2 Doubly singular limits (3/4)

Let's look at a three-mass triangle

```
oTriangle21 = LoadResults(settings.base_res_path + "6g_pmpmpm_G/triang
```

```
# Slope in lim ⟨3|(1+2)|4], Δ_135 → 0↔
```

```
Consistency check:
The largest momentum violation is 4.97541640258e-308
The largest on shell violation is 3.56011817361e-307
Δ_135 = 1e-60
⟨3|(1+2)|4] = 1e-30
Π_351 = 4.28812278569e-15
Ω_351 = 5.17028345077e-14
...
The slope in this limit is: 6.5. Need square roots?
```

2.2 Doubly singular limits (4/4)

All branch cuts should have been taken care of by generalised unitarity cuts.

We should be able to explain this behaviour without introducing square roots.

```
# 4 Δ_135 = Π_351 ^ 2 + 4 ⟨3|(1+2)|4] ⟨4|(1+2)|3]↔
```

```
(-94985.9529022-277600.460344j)
(-94985.9529022-277600.460344j)
```

```
# Π_351 = s_123 - s_124
print(gmpTools.to_complex(oParticles.compute("s_123") -
                          oParticles.compute("s_124")))
print(gmpTools.to_complex(oParticles.compute("Π_351")))
```

```
(-635.655095366+881.193534264j)
(-635.655095366+881.193534264j)
```

3.1 Partial fraction decompositions (1/2)

Forbidden pairs	Forced pairs	Optional pairs
⟨12⟩, [12]: 1.0, 2 → 2	⟨12⟩, [45]: 2.0, 2 → 2	⟨12⟩, ⟨23⟩: 2.0, 30 → 5
⟨12⟩, ⟨34⟩: 1.0, 2 → 2		⟨12⟩, [34]: 2.0, 12 → 3
		⟨16⟩, [45]: 2.0, 12 → 3

```
# Degeneracy in 'cleaned' phase space
pprint(oUnknown.true_friends["⟨1|2⟩", "⟨2|3⟩"])
pprint(oUnknown.true_friends["⟨1|2⟩", "[3|4]"])
pprint(oUnknown.true_friends["⟨1|6⟩", "[4|5]"])
```

```
[⟨1|2⟩, ⟨2|3⟩, ⟨3|(1+2)|6], ⟨1|(2+3)|4], s_123]
[⟨1|(2+3)|4], ⟨1|2⟩, [3|4]]
[⟨1|6⟩, [4|5], ⟨1|(2+3)|4]]
```

3.1 Partial fraction decompositions (2/2)

We can now decompose our denominator into smaller pieces. This denominator ansatz can be generated automatically using information from collinear limits or inserted by hand.

```
# Difference in complexity↔
```

```
Least common denominator, w/ ansatz of O(10 000):
/⟨1|2⟩[1|2]⟨1|6⟩[1|6]⟨2|3⟩[2|3]⟨3|4⟩[3|4]⟨4|5⟩[4|5]⟨5|6⟩
[5|6]s_123s_234s_345
[16], [[-2, 2, -2, 2, -2, 2]]

After partial fractioning, w/ ansatz of length 15:
/⟨1|2⟩⟨2|3⟩[4|5][5|6]⟨1|(2+3)|4]⟨3|(1+2)|6]s_123
[8], [[0, 4, 0, 0, -4, 0]]
```

3.2 Fitting of generic ansatze (1/3)

The most generic ansatz for the given mass dimension and phase weights is built:

```
for entry in Ansatz([8], [[0, 4, 0, 0, -4, 0]])[0]:
    pprint("".join(entry))
```

```
Obtained ansatz from Daniel's spinor solve with lM, lPW:
[8], [[0, 4, 0, 0, -4, 0]]. Size: 15.
⟨1|2⟩⟨1|2⟩⟨1|2⟩⟨1|2⟩[1|5][1|5][1|5][1|5]
⟨1|2⟩⟨1|2⟩⟨1|2⟩⟨2|3⟩[1|5][1|5][1|5][3|5]
⟨1|2⟩⟨1|2⟩⟨1|2⟩⟨2|4⟩[1|5][1|5][1|5][4|5]
⟨1|2⟩⟨1|2⟩⟨2|3⟩⟨2|3⟩[1|5][1|5][3|5][3|5]
⟨1|2⟩⟨1|2⟩⟨2|3⟩⟨2|4⟩[1|5][1|5][3|5][4|5]
⟨1|2⟩⟨1|2⟩⟨2|4⟩⟨2|4⟩[1|5][1|5][4|5][4|5]
⟨1|2⟩⟨2|3⟩⟨2|3⟩⟨2|3⟩[1|5][3|5][3|5][3|5]
⟨1|2⟩⟨2|3⟩⟨2|3⟩⟨2|4⟩[1|5][3|5][3|5][4|5]
⟨1|2⟩⟨2|3⟩⟨2|4⟩⟨2|4⟩[1|5][3|5][4|5][4|5]
⟨1|2⟩⟨2|4⟩⟨2|4⟩⟨2|4⟩[1|5][4|5][4|5][4|5]
⟨2|3⟩⟨2|3⟩⟨2|3⟩⟨2|3⟩[3|5][3|5][3|5][3|5]
⟨2|3⟩⟨2|3⟩⟨2|3⟩⟨2|4⟩[3|5][3|5][3|5][4|5]
⟨2|3⟩⟨2|3⟩⟨2|4⟩⟨2|4⟩[3|5][3|5][4|5][4|5]
⟨2|3⟩⟨2|4⟩⟨2|4⟩⟨2|4⟩[3|5][4|5][4|5][4|5]
⟨2|4⟩⟨2|4⟩⟨2|4⟩⟨2|4⟩[4|5][4|5][4|5][4|5]
```

3.2 Fitting of generic ansatze (2/3)

The linear system of equations for the coefficients is then solved by numerical inversion.

```
# Chose inversion settings: cpu / gpu↔
```

```
# Fit the coefficients of the ansatz:
oTerms.fit_numerators();
# ... lot of information gets printed ...
```

```
Time elapsed in row reduction:  0.00126004219055 .
Iteration number 1: dropped_redundant: 0, dropped_zero: 1
0, dropped_total: 10.
Coeff. of ⟨1|2⟩⟨1|2⟩⟨1|2⟩⟨1|2⟩[1|5][1|5][1|5][1|5]: 1*I
Coeff. of ⟨1|2⟩⟨1|2⟩⟨1|2⟩⟨2|3⟩[1|5][1|5][1|5][3|5]: -4*I
Coeff. of ⟨1|2⟩⟨1|2⟩⟨2|3⟩⟨2|3⟩[1|5][1|5][3|5][3|5]: 6*I
Coeff. of ⟨1|2⟩⟨2|3⟩⟨2|3⟩⟨2|3⟩[1|5][3|5][3|5][3|5]: -4*I
Coeff. of ⟨2|3⟩⟨2|3⟩⟨2|3⟩⟨2|3⟩[3|5][3|5][3|5][3|5]: 1*I
This piece correctly removes the singularity ([0])

Refining the fit...
The least common denominator is
⟨2|(1+3)|5]⁴/⟨1|2⟩⟨2|3⟩[4|5][5|6]⟨1|(2+3)|4]⟨3|(1+2)|6]s_
```

3.2 Fitting of generic ansatze (3/3)

Hence the result for $A(1_g^+ 2_g^- 3_g^+ 4_g^- 5_g^+ 6_g^-)$ tree amplitude:

```
print(oTerms)
```

```
+1I⟨2|(1+3)|5]⁴/⟨1|2⟩⟨2|3⟩[4|5][5|6]⟨1|(2+3)|4]⟨3|(1+2)|
6]s_123
(u'165432', False)
(u'216543', True)
```

(165432, False) means: 123456 → 165432

(216543, True) means: 123456 → 216543 + swap all angle and square brackets

A less trivial result follows.

$A_R^{1-loop}(1_g^+ 2_g^- 3_g^+ 4_g^- 5_g^+ 6_g^-)$

```
# Check analytical result (displayed is difference to numerical)↔
```

```
(2.80569172739e-299+9.0050121666e-300j)
```

```
RationalPDF # Showing only first few terms
```

$$\frac{2/3i\langle12\rangle^3[15]^3[23]s_{123}}{[45]\langle1|2+3|1]^2\langle1|2+3|4]\langle1|2+3|6]\langle3|1+2|6]}+$$

$$\frac{-2/3i\langle12\rangle^3[15]^3[23]\langle3|1+2|5]}{\langle13\rangle[45][56]\langle1|2+3|1]^2\langle1|2+3|4]\langle3|1+2|6]}+$$

$$\frac{1i\langle12\rangle^3[15]^2\langle23\rangle[23]^2[56]}{[45]\langle1|2+3|1]\langle1|2+3|4]\langle1|2+3|6]^2\langle3|1+2|6]}+$$

$$\frac{\langle12\rangle^2[15]^2[23](-1i\langle12\rangle[15]+2i\langle23\rangle[35])}{[45]\langle1|2+3|1]\langle1|2+3|4]\langle1|2+3|6]\langle3|1+2|6]}+$$

$$\frac{1i\langle12\rangle^3[15]^2[25]\langle3|1+2|5]}{\langle13\rangle^2[45][56]\langle1|2+3|1]\langle1|2+3|4]\langle3|1+2|6]}+$$

$$\frac{2i\langle12\rangle^2[15]^2[35]\langle3|1+2|5]}{\langle13\rangle[45][56]\langle1|2+3|1]\langle1|2+3|4]\langle3|1+2|6]}+$$

$$\frac{-1i\langle12\rangle^2[15]\langle23\rangle[25]^2\langle2|1+3|5]}{\langle13\rangle^2[45][56]\langle1|2+3|4]\langle3|1+2|6]s_{123}}+$$

$$\frac{-1i\langle12\rangle^2[15]^2[25]\langle2|1+3|5]}{\langle13\rangle[45][56]\langle1|2+3|4]\langle3|1+2|6]s_{123}}+$$

Thank you!

Questions?