The CHY formalism for massless scattering

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Abstract

The Cachazo-He-Yuan (CHY) formalism describes interactions among sub-atomic particles and allows the computation of scattering amplitudes. It is equivalent to, and at the same time fundamentally different from, the perturbative treatment of quantum field theory using Feynman diagrams (up to tree-level). It deals in particular with the scattering of n massless particles in an arbitrary D-dimensional flat space-time. This is achieved by a map from momentum space to the Riemann sphere with punctures. Starting from this map, we discuss the so-called Scattering Equations, the proof for their polynomial form by Dolan and Goddard, and their general solution in terms of the determinant of a $(n-3)! \times (n-3)!$ matrix. A program in Mathematica to perform these involved calculations for general n is given as well. Finally, we briefly discuss of how the scattering amplitudes can be obtained.

Contents

1. General Introduction

At the beginning of the last century important scientific results were obtained by relying on quantization procedures, the most famous probably being Plank's solution to the black-body radiation problem in 1900, Einstein's 1905 paper on the photoelectric effect, and Bohr's 1913 model of the atom. Quantum mechanics was born. Since then, the need to extend the quantization to the theory of fields become evident. Over the years, work done by many important physicists (Dirac, Fermi, Feynman, Fock, Pauli, and others) led to the development of quantum field theories (QFT), such as quantum electrodynamics (QED) and quantum chromodynamics (QCD). These are the backbone of the Standard Model, which constitutes our best, experimentally tested, understanding of physics at the subatomic scale.

QFTs can be discussed in roughly three steps.
First of all, a sensible starting point is classical field theories [1]. Their fundamental object is the Lagrangian density. The equations of motion of the field are obtained by minimizing the action, defined as an integral of the Lagrangian density over space-time. Further important elements are the currents and the conserved charges corresponding to symmetries of the Lagrangian (Noether's theorem).
As a brief reminder, we can consider classical electrodynamics. The Lagrangian is given by:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\nu \text{ with } F_{\mu\nu} = (\partial_\mu A_\nu) - (\partial_\nu A_\mu),$$

where A^μ and J^ν are 4-vector defined as $A^\mu = (\phi, \boldsymbol{A})$, with ϕ and $\mathbf{A}$ the usual scalar and vector potentials, and $J^\nu = (\rho, \boldsymbol{j})$, with ρ the charge density and $\boldsymbol{j}$ the current. Minimizing the action yields the Euler-Lagrange equations, which, in this case, are indeed Maxwell's equations (two of the four):

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

The other two are given by the Bianchi's identity:

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0.$$

Noether's theorem applied to space-time translations yields the energy-momentum tensor (ignoring the source term $A_\mu J^\nu$):

$$T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}.$$

Secondly, the quantization of classical field theories is usually done via a Canonical (or Second) Quantization [2]. It requires to translate the Lagrangian into an Hamiltonian, and then to impose canonical commutation relations to the fields and momenta, or to certain coefficients, which are promoted to creation and annihilation operators. Particles are then nothing more than excitations (or quanta) of these fields.
Finally, the quantities we want to compute are probability amplitudes for scattering events [3],[4]. For instance, we may want to know the probability of obtaining quark-antiquark pairs by electron-positron annihilation: $e^+e^- \to \gamma \to q\bar{q}$. The mathematical formula for the amplitude, as written in [3], is:

$$\mathcal{A} = \lim_{t_\pm \to \pm\infty} \langle f| U(t_+, t_-) |i\rangle = \langle f| S |i\rangle,$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states, and $U(t_+, t_-)$ is the propagator (which describes the probability of going from an initial state at time t_- to a final state at t_+). In the above limit, the propagator is usually written with just the letter S and is known as the S-Matrix. This matrix element will probably be familiar to the reader; for example, it appears in Fermi's Golden Rule for transition rates.
Scattering amplitudes are usually computed using Feynman's diagrams and integration rules. However, this type of computation can easily become very complex, especially when a large number of diagrams are involved. Therefore, it is important for the advancement of the field to give serious consideration to alternative - but equivalent - formalisms, for they may provide new insights, allow easier computations, or be more easily generalized to new theories.

In this report we discuss the formalism recently introduced by Cachazo, He and Yuan [CHY] [5]-[9] for the computation of tree level scattering amplitudes of massless particles in a flat space-time of arbitrary dimensions. We will not try to show the logical or mathematical equivalence between this formalism and the standard theory briefly presented above, since this would require us to directly discuss string theories. Instead, we will develop the formalism and show that it yields the correct expressions for the scattering amplitudes.
We will focus in particular on the aspect of the formalism that appears to be generalizable to a number of different theories, that is the so called Scattering Equations [SE] and their solutions. A version of these equations was first discovered in the 1970s by Fairlie and Roberts [10]-[12] in the context of emerging string theories and rediscovered later by Gross and Mende [13] and CHY.

The Scattering Equations encode the kinematics of the scattering process in a fundamentally different fashion compared to the standard quantum field theory. Whereas QFT generalizes the classical concepts of action, Lagrangian and Hamiltonian via a Canonical Quantization as we have discussed above, the SE are based on a map from momentum space to the Riemann sphere[1] and the thereby defined punctures (special points on the sphere). Unfortunately, working in this complex space implies that the physical meaning of intermediate results is rather obscure. However, the scattering amplitudes - which are the real quantities we are interested in as physicists, since they are the observables - have been shown to nicely reproduce the correct results for several theories. For instance, the correspondence to Yang-Mills tree amplitudes has been proven by Dolan and Goddard in [14].

In this report, on the one hand, we specialize to massless particles because understanding the CHY formalism in this limit is essential before moving on to the massive case. In fact, the latter was investigated as a generalization of the massless limit and is considerably more mathematically involved (for instance, eq. (3.1) would have be to replaced by eq. (1.8) of reference [14]). On the other hand, our discussion is valid for particles of arbitrary spin, colour, and charge, since these properties of the particles are not reflected by the SE (they are instead encoded in $\hat{\Psi}$, see eq (6.1)). As a consequence, by the end of the report, we have almost all elements necessary to compute amplitudes involving real-world processes like a three gluon interaction (Figure 1).

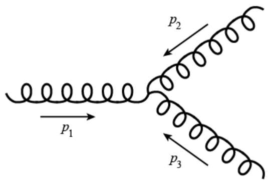

Figure 1: Three gluon Feynman dagram.

What we do compute explicitly in section 6 via a contour integral encircling the solutions of the SE is the massless ϕ^3-theory, 4-particles scattering amplitude. The relevant Feynman diagrams for this process are given in Figure 2.

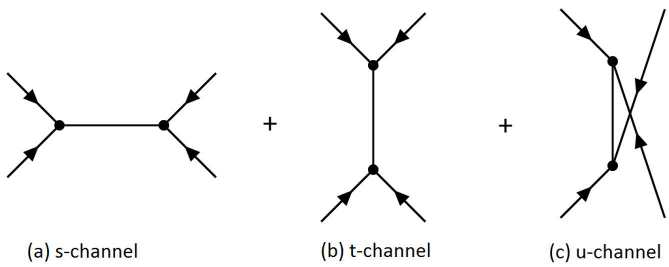

Figure 2: Feynman diagrams for the massless ϕ^3-theory, 4-particles scattering amplitude.

The computation of the scattering amplitude following the Feynman rules is rather straightforward (for instance, see section 1.5 of [3]) and indeed both approaches yield the same expression (eq. (6.2)).

[1]Complex plane plus the point at infinity, which is topologically a sphere - see Appendix I.

The structure of this report is hereby outlined.
In section 2 we will introduce notation and the key relation between the momenta and the punctures (eq. (2.1)). This will constitute our starting point and as such will be taken as given. All other results up to the end of section 5 will be explicitly derived from this one, together with basic assumptions on the momenta. In this section we also give a proof for the expression of the polynomial in the numerator of the integrand in equation (2.1) - see Appendix II for details.
This result is then used in section 3 to prove the CHY expression for the Scattering Equations (eq. (3.1)) - algebra in Appendix III. The results of these proofs can be found in the references (more specifically at the beginning of [6]), but the proofs themselves here presented were left as exercises for the readers. In section 3.1 we show that the SE are Möbius invariant (Appendix IV), and in section 3.2 we investigate the presence - or, as a matter of fact, the absence - of additional symmetries to the SE (Appendix V).
In section 4 we discuss how the SE can be recast into polynomial form, as shown by Dolan and Goddard in [15]. An explanation of the quantitative aspects of their reasoning can be found in section VI of the Appendix.
Then, in section 5 we explain how the SE can be solved using an elimination algorithm, as presented in another paper by DG [16] and in a similar paper by Cardona and Kalousios [17]. We wrote a program in Mathematica based on this elimination algorithm, and then we improved it by implementing a faster recursion algorithm, similar to that of CK. The program itself and sample outputs can be found in section VII of the Appendix.
In section 6 we briefly discuss how the scattering amplitudes can be obtained by integrating over the solutions to the SE and accounting for permutations. In particular, as an example, we look at the simplest possible theory, that is scalar ϕ^3 theory.
Lastly, conclusions and final comments are given in the section 7.

2. Mathematical Introduction and Notation

In this section we will establish the mathematical notation - largely following that of CHY - and explain origin and properties of key equations.

The subject of our analysis will be the scattering of n massless particles in $D-1$ spatial dimensions. In analogy to special relativity 4-vectors, the n particles momenta will be k_a^μ, with $a \in A$,

$$A = \{1, ..., n\},$$

indexing the particle and $\mu \in \{0, 1, ..., D-1\}$ indexing the space-time components. The inner product is defined as $k^2 = k^\mu k_\mu = k^\mu \eta_{\mu\nu} k^\nu$, with $\eta_{\mu\nu}$ a generalized Minkowski metric $diag(+1, -1, ..., -1)$.

Two basic properties we require are the massless and the null conserved momenta conditions, respectively given by:

$$k_a^2 = 0 \;\; \forall a \in A \qquad \text{and} \qquad \sum_{a \in A} k_a^\mu = 0 \;\; \forall \mu.$$

The first condition is simply $E^2 - p^2 = m^2 = 0$ for massless particles. The second condition is slightly trickier. We are claiming that the sum of each component of D-momentum over all the n particles is zero. Usually we would have a set of incoming and a set of outgoing particles (Figure 3 (a)). For this situation we require $\sum_{a \in incoming} k_a^\mu = \sum_{b \in outgoing} k_b^\mu$, that is energy-momentum conservation. However, we may as well set all the particles to, say, be going in (Figure 3 (b)). Then the above equation reads $\sum_{a \in all\ particles} k_a^\mu = 0$, where we understand that particles with negative energy ($\mu = 0$ component) need to be taken on the other side of the equal sign and interpreted as particles with positive energy leaving the scattering process. This is completely analogous to Kirchhoff's current law in circuit theory.

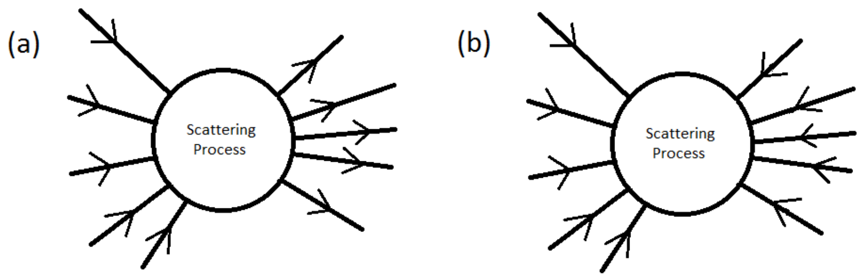

Figure 3: (a) Particles coming in and going out, all with positive energy.
(b) All particles coming in. Negative energy particles are understood to be outgoing particles with positive energy.

Let us now define the Riemann sphere (Σ) as the one-point compactification of the complex plane ($\mathbb{C}$) with the point at complex infinity ($\tilde{\infty}$): $\Sigma \equiv \mathbb{C} \cup \tilde{\infty}$ (For further details on the Riemann sphere see Appendix I). The n punctures - that is the n special points on the sphere that are related to the particles' momenta - are given the symbol σ_a, with $a \in A$.
It has been shown in [5] that the relation between the scattering data and the punctures is given by:

$$k_a^\mu = \frac{1}{2\pi i} \oint_{|z-\sigma_a|=\epsilon} dz \frac{p^\mu(z)}{\prod_{b\in A}(z-\sigma_b)} \ , \tag{2.1}$$

where $p^\mu(z)$ are D polynomials with coefficients depending on the momenta (k) and the punctures (σ). Equation (2.1) can be viewed as sets of n equations for each of these polynomials. If we assume all the σ's to be distinct and $p^\mu(z)$ to be non singular at the pole, then - using Cauchy's residue theorem - we obtain:

$$k_a^\mu = \frac{1}{\cancel{2\pi i}} \times \cancel{2\pi i} \times \underset{z\to\sigma_a}{Res} = \frac{p^\mu(\sigma_a)}{\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)} \ ,$$

and therefore:

$$p^\mu(\sigma_a) = k_a^\mu \prod_{\substack{b\in A\\ b\neq a}} (\sigma_a - \sigma_b) \ . \tag{2.2}$$

We will now introduce the following notation, as in [6]. Define $\{\sigma\}_b^m$ as the symmetrized product of m σ's which do not involve σ_b:

$$\{\sigma\}_b^m \equiv (-1)^m \sum_{\{a_i\}\subseteq A\setminus\{b\}} \sigma_{a_1}\sigma_{a_2}...\sigma_{a_m} \ . \tag{2.3}$$

Then, we may rewrite eq. (2.2) as:

$$p^\mu(\sigma_a) = k_a^\mu \sum_{m=0}^{n-1} \sigma_a^m \{\sigma\}_a^{n-1-m} \ , \tag{2.4}$$

where we are also defining $\{\sigma\}_a^0$ to be 1. Since eq. (2.4) must hold for any a, we would like to rewrite it in a way that makes the coefficients of σ_a^m explicitly symmetric in the punctures σ's and the momenta k's. Such an expression is given by (proof in Appendix II):

$$p^\mu(\sigma_a) = \sum_{b\in A} k_b^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_b^{n-1-m} \ . \tag{2.5}$$

We can then conclude from eq. (2.5) that the general expression for $p^\mu(z)$ must be given by:

$$p^{\mu}(z) = \sum_{m=0}^{n-2} \sum_{b\in A} k_b^{\mu} z^m \{\sigma\}_b^{n-1-m} \ . \tag{2.6}$$

This is exactly what one obtains by combining equations (2.3) and (2.4) in reference [6].

3. The Scattering Equations

Thus far, we have shown that $p^\mu(z)$ are D degree $n-2$ polynomials and we have found their coefficients in terms of the punctures and the momenta. However, the punctures themselves are defined by eq. (2.1) and, therefore, the next step is to find the equations relating the punctures to the momenta. These are the so called Scattering Equations. They encode the kinematics of the scattering process into the language of the punctured Riemann sphere. The solutions of the SE allow to compute the scattering amplitudes. Since the momenta appear only in dot products, the Lorentz invariance is automatically guaranteed.

The SE are most naturally obtained by translating the massless conditions $k_a^2 = 0$ to conditions on the polynomials $p^\mu(z)$, in particular eq. (2.1) requires $p(z)^2 = 0\ \forall z$. Since $p(z)^2$ is a polynomial of degree $2n-4$ and it is not monic, we require $2n-3$ (i.e. degree $+\,1$) independent conditions to specify it. n of these are found by evaluating $p(z)^2$ at each of the punctures. Equation (2.2) shows that these simply give $k_a^2 = 0$. The remaining $n-3$ conditions are what we are looking for and they are given by $\frac{d}{dz}p^2(z) = 0$, i.e. by $p(z)\cdot p'(z) = 0$. Evaluating this last condition at the punctures, i.e. $p(\sigma_a)\cdot p'(\sigma_a) = 0$, leads to the Scattering Equations (details in Appendix III):

$$f_a(\sigma, k) \equiv \sum_{\substack{b\in A\\ b\neq a}} \frac{k_a\cdot k_b}{\sigma_a - \sigma_b} = 0, \qquad \forall a \in A\ . \tag{3.1}$$

We can show that of these n equations only $n-3$ are independent by looking at the symmetry of the punctures. We must expect the σ's not to be completely fixed by the Scattering Equations because for the polynomial $p(z)^2$ to be zero $\forall z$ it must be effectively in more variables than just z, otherwise it would have at most as many solutions as its degree $(2n-4)$.

3.1 The Möbius Invariance

The coordinates of the Riemann sphere under a Möbius transformation transform as follows:

$$z \to \zeta = \frac{\alpha z + \beta}{\gamma z + \delta}\ , \tag{3.2}$$

where the four complex parameters α, β, γ and δ, in order to avoid cancellation between numerator and denominator, must also satisfy the following condition:

$$\frac{\alpha(z+\beta/\alpha)}{\gamma(z+\delta/\gamma)} \quad \to \quad \beta/\alpha \neq \delta/\gamma \quad \text{or} \quad \alpha\delta - \beta\gamma \neq 0\ .$$

Showing that the Scattering Equations are invariant under this transformation is quite simple (cf. equation (1.5) of reference [15], for instance). For completeness we have reproduced the proof in details in Appendix IV.

In the literature it is customary to partially fix the Möbius Invariance. In particular, it is convenient to let $\sigma_1 = \infty$ and $\sigma_n = 0$. We can understand this geometrically as fixing the north and south pole of the Riemann sphere. Rewrite equation (3.2) as:

$$\frac{\not{\delta}(\frac{\alpha}{\delta}\sigma_a + \frac{\beta}{\delta})}{\not{\delta}(\frac{\gamma}{\delta}\sigma_a + 1)} \quad -\text{renaming}\to \quad \frac{\alpha\sigma_a + \beta}{\gamma\sigma_a + 1}\ .$$

Then, $\sigma_1 = \infty$ and $\sigma_n = 0$ respectively force $\gamma = 0$ and $\beta = 0$. This leaves us with: $\sigma_a \to \alpha\sigma_a$. By writing $\alpha = r_\alpha e^{i\phi_\alpha}$ it is clear that the Möbius transformation consists now of just a rotation and/or scaling of the Riemann sphere.

Although we have shown that the Scattering Equations are invariant under Möbius Transformation, we have not mathematically shown that this is their full symmetry. It does seem to be the case, since we were looking for $2n-3$ conditions and we have found that many, but a rigorous proof would be nice nonetheless.

3.2 The Full Symmetry Problem

An additional symmetry in the Scattering Equations would lower the number of independent ones below $n-3$ and thus require us to look somewhere else for the remaining conditions. We have already set $p(z)^2|_{z=\sigma_a} = 0$ and $\frac{d}{dz}p(z)^2|_{z=\sigma_a} = 0$, therefore it would be natural to look at $\frac{d^2}{dz^2}p(z)^2|_{z=\sigma_a} = 0$ next. This is indeed the case:

$$\frac{d^2}{dz^2}p(z)^2|_{z=\sigma_a} = 0 \quad \to \quad p'(\sigma_a)^2 + p(\sigma_a)\cdot p''(\sigma_a) = 0 \; .$$

Our analysis (see Appendix V) shows that the second derivative of $p(z)^2$ vanishes identically provided both the Scattering Equations and the basic assumptions on the momenta are satisfied. Therefore, it would seem that it is not possible to obtain additional conditions for the punctures, meaning that Möbius should be the full symmetry. This is actually implicitly proven by fixing 3 punctures and finding $(n-3)!$ solutions for the remaining ones, as in [6] and [15].

Indeed, the next step would be to solve the Scattering Equations. However, to do so with the form given by equation (3.1) is rather impractical. In fact, once we eliminate the denominators by multiplying by $\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a - \sigma_b)$ we obtain a system of n equations, each of which is of much higher order in the σ's than necessary.

4. The Scattering Equations in Polynomial Form

A solution to this problem was found by Dolan and Goddard in reference [15]. They managed to recast the Scattering Equations to a system of $n-3$ homogeneous equations in the punctures, using their notation:

$$h_m = \sum_{\substack{S\subset A'\\ |S|=m}} k_{S_1}^2 \sigma_S = 0 \quad \text{with} \quad A' = A/\{1,n\} \text{ and } S_1 = S\cup\{1\} \; , \tag{4.1}$$

where k_S and σ_S are defined as follows:

$$k_S = \sum_{b\in S} k_b \quad \text{and} \quad \sigma_S = \prod_{b\in S}\sigma_b \; .$$

In this expression each h_m is of order m in the σ's and has the additional nice feature of being linear in each puncture taken separately. What follows is an explanation of DG's reasoning to derive eq. (4.1) from eq. (3.1). Since this requires some involved algebra, the mathematics is presented in Appendix VI. The proof is mainly as in [15], with some slight changes in a couple of steps, mostly to keep things as easy and clear as possible.

The existence of h_m's is suggested, if not guaranteed, by the combination of two facts. The first one is that the algorithm presented by CHY in section 3 of reference [6] proves that the number of solutions to the Scattering Equations is $(n-3)!$. The second one is that Bézout's theorem states that the number of solutions to a system of polynomials is bounded by the product of their degrees and that this bound may be attained if the equations are suitably written. Therefore, not only h_m's are of much lower degree than f_a's, but they are also as low as they can get.

The first step to obtain (4.1) from (3.1) is to prove that the Scattering Equations $f_a = 0$, $a \in A$ are equivalent to $g_m = 0$, $2 \le m \le n-2$, with g_m defined as:

$$g_m(\sigma, k) = \sum_{a \in A} \sigma_a^{m+1} f_a(\sigma, k) \; . \tag{4.2}$$

This is the first proposition proven in Appendix VI.

The next step is to introduce the following polynomials:

$$\tilde{h}_m = \sum_{\substack{S \subset A \\ |S|=m}} k_S^2 \sigma_S = 0 \; , \tag{4.3}$$

and prove that $\tilde{h}_m = 0$ are equivalent to $f_a = 0$ by proving that $\tilde{h}_m = 0$ and $g_m = 0$ are equivalent. This requires more involved algebra and it is the second proposition proven in Appendix VI.

At this point the h_m's are obtained from the $\tilde{h}_m$'s fixing the Möbius invariance by letting $\sigma_1 \to \infty$ and $\sigma_n \to 0$, as discussed before. The consequence of this is that in equation (4.3) only the subsets S of A including the element {1} and not the element {n} will survive. Formally:

$$h_m = \lim_{\sigma_1 \to \infty} \frac{\tilde{h}_{m+1}|_{\sigma_n = 0}}{\sigma_1} = \sum_{\substack{S \subset A' \\ |S|=m}} k_{S_1}^2 \sigma_S = 0 \quad \text{with} \quad A' = A/\{1, n\} \text{ and } S_1 = S \cup \{1\} \; .$$

We are now in a much better position to solve these equations.

5. Solving the Scattering Equations

The work presented here is based on two very recent and relatively similar papers: [16] by Dolan and Goddard and [17] by Cardona and Kalousios.
We follow their reasoning to obtain a $(n-3)! \times (n-3)!$ square matrix, whose determinant provides an homogeneous equation in σ_{n-1} and σ_{n-2}, of order $(n-3)!$ and independent of all the other σ's. This can be viewed as an equation for $\sigma_{n-2}/\sigma_{n-1}$ and provides the foundation of the solution.
We wrote a program in Mathematica based on a recursive algorithm, similar to that of CK, to generate this determinant for arbitrary n. It can be found, together with sample outputs, in Appendix VII.

It is extremely useful to consider a couple of examples in order to recognize the underlining pattern before moving on to the general case.
n = 4: this is trivial, we have one equation and it is the equation we are looking for:

$$h_1 = \sigma_2 k_{\{1,2\}}^2 + \sigma_3 k_{\{1,3\}}^2 = 0 \; .$$

n = 5: we now have two equations in three knowns:

$$\begin{aligned} h_1 &= \sigma_2 k_{\{1,2\}}^2 + \sigma_3 k_{\{1,3\}}^2 + \sigma_4 k_{\{1,4\}}^2 \; , \\ h_2 &= \sigma_2 \sigma_3 k_{\{1,2,3\}}^2 + \sigma_2 \sigma_4 k_{\{1,2,4\}}^2 + \sigma_3 \sigma_4 k_{\{1,3,4\}}^2 \; . \end{aligned}$$

We wish to eliminate σ_2. In order to do so, define:

$$H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad \text{and} \qquad H^{\sigma_2} = \partial_{\sigma_2} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \; .$$

Then we may rewrite the Scattering Equations as:

$$\left(\begin{array}{cc} H & H^{\sigma_2} \end{array} \right)|_{\sigma_2=0} \cdot \left(\begin{array}{c} 1 \\ \sigma_2 \end{array} \right) = \left(\begin{array}{cc} \sigma_3 k^2_{\{1,3\}} + \sigma_4 k^2_{\{1,4\}} & k^2_{\{1,2\}} \\ \sigma_3\sigma_4 k^2_{\{1,3,4\}} & \sigma_3 k^2_{\{1,2,3\}} + \sigma_4 k^2_{\{1,2,4\}} \end{array} \right) \cdot \left(\begin{array}{c} 1 \\ \sigma_2 \end{array} \right) = 0 \,,$$

which is satisfied if the determinant vanishes. Explicitly:

$$\sigma_3^2 k^2_{\{1,3\}} k^2_{\{1,2,3\}} + \sigma_4\sigma_3 \left(k^2_{\{1,4\}} k^2_{\{1,2,3\}} + k^2_{\{1,3\}} k^2_{\{1,2,4\}} - k^2_{\{1,2\}} k^2_{\{1,3,4\}} \right) + \sigma_4^2 k^2_{\{1,4\}} k^2_{\{1,2,4\}} = 0 \,,$$

giving the required second order homogeneous equation in σ_3 and σ_4. We may then find σ_2/σ_4 for each of the two solutions using $h_1 = 0$.
A final comment on this case, as one sees this reasoning requires us to evaluated the (H, H^{σ_2}) matrix at $\sigma_2 = 0$. However, by adding $\sigma_2 H^{\sigma_2}$ to the first column, we obtain the same matrix not evaluated at $\sigma_2 = 0$. Since we are taking a determinant we can conclude that setting $\sigma_2 = 0$ is not necessary, and indeed this result has been shown by DG in [16] to generalize to all n. However, since calculations become quickly cumbersome as n increases it may be convenient to set σ's to zero whenever possible.

<u>n = 6</u>: the Scattering Equations now are:

$$\begin{aligned} h_1 &= \sigma_2 k^2_{\{1,2\}} + \sigma_3 k^2_{\{1,3\}} + \sigma_4 k^2_{\{1,4\}} + \sigma_5 k^2_{\{1,5\}} \,, \\ h_2 &= \sigma_2\sigma_3 k^2_{\{1,2,3\}} + \sigma_2\sigma_4 k^2_{\{1,2,4\}} + \sigma_3\sigma_4 k^2_{\{1,3,4\}} + \sigma_2\sigma_5 k^2_{\{1,2,5\}} + \sigma_3\sigma_5 k^2_{\{1,3,5\}} + \sigma_4\sigma_5 k^2_{\{1,4,5\}} \,, \\ h_3 &= \sigma_2\sigma_3\sigma_4 k^2_{\{1,2,3,4\}} + \sigma_2\sigma_3\sigma_5 k^2_{\{1,2,3,5\}} + \sigma_2\sigma_4\sigma_5 k^2_{\{1,2,4,5\}} + \sigma_3\sigma_4\sigma_5 k^2_{\{1,3,4,5\}} \,. \end{aligned}$$

We wish to eliminate σ_2 and σ_3. To achieve this we may use a mathematical technique known as *Elimination Theory*, as it is done in both [16] and [17]. The idea is to introduce new variables and new equations until the system is over-specified and yield a consistency condition in the form of $\det(M) = 0$, as in the $n = 5$ case.

In this case we wish to eliminate 4 variables (including "1"): $\{1, \sigma_2\} \otimes \{1, \sigma_3\} = \{1, \sigma_2, \sigma_3, \sigma_2\sigma_3\}$, but only have 3 equations: $\{h_1, h_2, h_3\}$. Therefore we introduce another factor of $\{1, \sigma_3\}$ and obtain a new set of 6 variables: $V^T = \{1, \sigma_3\} \otimes \{1, \sigma_2, \sigma_3, \sigma_2\sigma_3\} = \{1, \sigma_2, \sigma_3, \sigma_2\sigma_3, \sigma_3^2, \sigma_2\sigma_3^2\}$, and a new set of 6 equations: $\{1, \sigma_3\} \otimes \{h_1, h_2, h_3\} = \{h_1, h_2, h_3, \sigma_3 h_1, \sigma_3 h_2, \sigma_3 h_3\}$. Now that we have as many variables as equations, we can obtain the consistency condition $det(M) = 0$, since:

$$M \cdot V = \left(\begin{array}{cccccc} H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 \\ 0 & 0 & H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} \end{array} \right) \cdot \left(\begin{array}{c} 1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_2\sigma_3 \\ \sigma_3^2 \\ \sigma_2\sigma_3^2 \end{array} \right) = 0 \,.$$

As for the $n = 5$ case, M can be understood as the matrix of coefficients (setting $\sigma_2, \sigma_3 = 0$) and multiplying it by the vector of variables V gives back the set of 6 equations given above. Note that the first line of M gives the sub set $\{h_1, h_2, h_3\}$, which does not depend on $\{\sigma_3^2, \sigma_2\sigma_3^2\}$, and the second line give $\{\sigma_3 h_1, \sigma_3 h_2, \sigma_3 h_3\}$, independent of $\{1, \sigma_2\}$. Also, it should be clear that now $H^T = (h_1, h_2, h_3)$.

<u>General n</u>: In general we have $n - 3$ equations, h_m $1 \leq m \leq n - 3$, in $n - 2$ unknowns, σ_i $2 \leq i \leq n - 1$. The original set of 2^{n-4} variables we wish to eliminate using elimination theory is given by: $V^T = \{1, \sigma_2\} \otimes \{1, \sigma_3\} \otimes ... \otimes \{1, \sigma_{n-3}\}$. We introduce a set $W^T = \{1\} \otimes \{1, \sigma_3\} \otimes \{1, \sigma_4, \sigma_4^2\} \otimes ... \otimes \{1, \sigma_{n-3}, ..., \sigma_{n-3}^{n-5}\}$, which contains $(n-4)!$ terms. The new set of variables is then given by $V^T \to V^T \otimes W^T = \{1, \sigma_2\} \otimes \{1, \sigma_3, \sigma_3^2\} \otimes ... \otimes \{1, \sigma_{n-3}, ..., \sigma_{n-3}^{n-4}\}$, of length $(n-3)!$. Similarly, the new $(n-3)!$ equations are given by $H^T \to H^T \otimes W^T$. Then, taking partial derivatives of the entries of H

w.r.t. those of V we construct the $(n-3)! \times (n-3)!$ matrix M whose determinant is the required equation.

Although this is the underlining reasoning, it is quite inefficient, even for a computer program. Instead, we can realize that each successive matrix, call it M_n, contains the previous one, call it M_{n-1}, repeated in some order and combined with zeros. Therefore, we can write a recursive relation to obtain the desired matrix. This is analogous to what Cardona and Kalousios do in [17]. An important point that we would like to stress here is that this approach allows us not to worry at all about what new variables and equations we are introducing, making the computation a lot faster. Of course a proof for this algorithm would have its foundation in the structure of the successive tensor products discussed above. With our notation we have:

$$M_i = \begin{pmatrix} M_{i-1} & M_{i-1}^{\sigma_{i-3}} & 0 & 0 & \dots & 0 & 0 \\ 0 & M_{i-1} & M_{i-1}^{\sigma_{i-3}} & 0 & \dots & 0 & 0 \\ 0 & 0 & M_{i-1} & M_{i-1}^{\sigma_{i-3}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & M_{i-1} & M_{i-1}^{\sigma_{i-3}} \end{pmatrix}, \quad M_4 = H, \quad H = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-3} \end{pmatrix},$$

with M_i of dimensions $(i-4) \times (i-3)$ when written in terms of M_{i-1}. For example, take $n = 6$:

$$M_6 = \begin{pmatrix} M_5 & M_5^{\sigma_3} & 0 \\ 0 & M_5 & M_5^{\sigma_3} \end{pmatrix}, \quad M_5 = \begin{pmatrix} M_4 & M_4^{\sigma_2} \end{pmatrix}, \quad M_4 = H, \quad H = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},$$

which agrees with our discussion in the examples above.

An important feature of this matrix which allows to make the computation faster is that each set of $(n-3)$ rows (when written explicitly in terms of the h_i's) is a cyclic permutation of the first. Therefore, computing the first $(n-3)$ rows gives the full matrix (after appropriate permutations).

Finally, the remaining σ's may be found by reintroducing them one at a time in the matrix and eliminating the appropriate column. For instance, say we would like to find σ_2, then all that we need to do is add $\sigma_2 \times$ second column to, say, the first column and then remove the second column. Note how this procedure leaves the system of equations (written as matrix of coefficients $\times$ vector of variables) unchanged. Removing any row, say the last one, makes the matrix square again. σ_2 is then obtained by setting the determinant of this new matrix to zero. Indeed, following this procedure for $n = 5$ gives back $h_1 = 0$ (see the $n = 5$ example in the previous page).

6. Scattering Amplitudes

We will now show how performing an integral over the solutions to the Scattering Equations yields the tree level scattering amplitudes. In [6] CHY proposed and presented evidence for an expression for the amplitudes. This formula was later proven by DG in [14] to be correct for Yang-Mills theories. Mathematically we have:

$$A_n = \int \hat{\Psi}_n(\sigma; k; \epsilon) \prod_{a \in A}{}' \delta(f_a(\sigma, k)) \prod_{a \in A} \frac{d\sigma_a}{(\sigma_a - \sigma_{a+1})^2} / d\omega \,,$$

or, more explicitly as a contour integral:

$$A_n = \oint_O \hat{\Psi}_n(\sigma; k; \epsilon) \prod_{a \in A}{}' \frac{1}{f_a(\sigma, k)} \prod_{a \in A} \frac{d\sigma_a}{(\sigma_a - \sigma_{a+1})^2} / d\omega \,, \tag{6.1}$$

where the contour O encircles all the non arbitrarily fixed, possibly complex, solutions to the Scattering Equations. The other symbols are defined as follows:

$$\prod_{a\in A}{}' = (\sigma_i-\sigma_j)(\sigma_j-\sigma_k)(\sigma_k-\sigma_i)\prod_{\substack{a\in A\\ a\neq i,j,k}} \ ,$$

$$d\omega = \frac{d\sigma_r d\sigma_s d\sigma_t}{(\sigma_r-\sigma_s)(\sigma_s-\sigma_t)(\sigma_t-\sigma_s)} \ .$$

In particular the latter is the Möbius group measure. We can intuitively understand that these definitions make the amplitude independent of the particular choice of punctures, since choosing $i, j, k = r, s, t$ and taking these 3 as the arbitrarily fixed punctures makes the integral over them drop out. Finally, the first term in the integral, $\hat{\Psi}$, depends on the particular theory we want to consider.

As an example, we will now compute the 4-particles amplitude for ϕ^3 theory. That is, the theory which is described in standard QFT language by the following Lagrangian:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (g/3!)\phi^3 \ .$$

For this theory we have $\hat{\Psi} = constant$. Also take $i, j, k = 1, 2, 4$ respectively and $\sigma_1 = \infty$, $\sigma_4 = 0$ (as done before) and $\sigma_2 = 1$. This completely fixes the Möbius invariance. Writing out equation (6.1) explicitly we obtain:

$$A_4^\phi \propto \oint_O \cancel{(\sigma_1-\sigma_2)}(\sigma_2-\sigma_4)\cancel{(\sigma_4-\sigma_1)}\frac{1}{f_3(\sigma,k)}\frac{\cancel{d\sigma_1}\cancel{d\sigma_2}d\sigma_3\cancel{d\sigma_4}}{\cancel{(\sigma_1-\sigma_2)^2}(\sigma_2-\sigma_3)^2(\sigma_3-\sigma_4)^2\cancel{(\sigma_4-\sigma_1)^2}}\frac{\cancel{(\sigma_1-\sigma_2)}(\sigma_2-\sigma_4)\cancel{(\sigma_4-\sigma_1)}}{\cancel{d\sigma_1}\cancel{d\sigma_2}\cancel{d\sigma_4}} \ ,$$

$$A_4^\phi \propto \oint_O \frac{(\sigma_2-\sigma_4)^2}{(\sigma_2-\sigma_3)^2(\sigma_3-\sigma_4)^2}\frac{1}{f_3(\sigma,k)}d\sigma_3 = \oint_O \frac{1}{(1-\sigma_3)^2(\sigma_3)^2}\frac{1}{f_3(\sigma,k)}d\sigma_3 \ .$$

Recall equation (3.1):

$$f_3(\sigma,k) = \frac{k_3\cdot k_1}{\sigma_3-\sigma_1} + \frac{k_3\cdot k_2}{\sigma_3-\sigma_2} + \frac{k_3\cdot k_4}{\sigma_3-\sigma_4} = \frac{k_3\cdot k_2}{\sigma_3-1} + \frac{k_3\cdot k_4}{\sigma_3} \ .$$

Hence:

$$A_4^\phi \propto \oint_O \frac{1}{(1-\sigma_3)\sigma_3(k_3\cdot k_2\sigma_3 + k_3\cdot k_4(\sigma_3-1))}d\sigma_3 \ .$$

We can now explicitly state the that the contour O encircles the value of σ_3 satisfying $f_3 = 0$:

$$k_3\cdot k_2\sigma_3 + k_3\cdot k_4(\sigma_3-1) = 0 \quad\Longrightarrow\quad \sigma_3 = \frac{k_3\cdot k_4}{k_3\cdot k_2 + k_3\cdot k_4} \ .$$

A quick sanity check can be done by looking back at section 5. There we found that for n=4:

$$h_1 = \sigma_2 k_{\{1,2\}}^2 + \sigma_3 k_{\{1,3\}}^2 = 0 \quad\Longrightarrow\quad \sigma_3 = -\frac{k_{\{1,2\}}^2}{k_{\{1,3\}}^2} = -\frac{k_1\cdot k_2}{k_1\cdot k_3} = \frac{k_3\cdot k_4}{k_3\cdot k_2 + k_3\cdot k_4} \ .$$

In the last step we need to use the null conserved momenta condition: $k_1 + k_2 + k_3 + k_4 = 0 \implies k_1\cdot k_2 = k_3\cdot k_4$ and $k_1\cdot k_3 = -(k_2+k_4)\cdot k_3$.

We can now write:

$$A_4^\phi \propto \oint_O \frac{1}{(1-\sigma_3)\sigma_3(k_3\cdot k_2 + k_3\cdot k_4)(\sigma_3 - \frac{k_3\cdot k_4}{k_3\cdot k_2+k_3\cdot k_4}))}d\sigma_3 \ ,$$

and using Cauchy's residue theorem:

$$A_4^\phi \propto \frac{1}{(1-\frac{k_3\cdot k_4}{k_3\cdot k_2+k_3\cdot k_4})\cancel{\frac{k_3\cdot k_4}{k_3\cdot k_2+k_3\cdot k_4}(k_3\cdot k_2+k_3\cdot k_4)}} = \frac{k_3\cdot k_2+k_3\cdot k_4}{k_3\cdot k_2\, k_3\cdot k_4} = \frac{1}{k_3\cdot k_2}+\frac{1}{k_3\cdot k_4}\ .$$

We can recast this in terms of the Mandelstam variables to obtain: $A_4^\phi \propto 1/s+1/t$.

The final result is obtained by accounting for the 6 permutations of $\{2,3,4\}$ and by reintroducing the appropriate coupling constant:

$$A_4^{\phi_{tot}} = \frac{g^2}{s}+\frac{g^2}{t}+\frac{g^2}{u}\ , \tag{6.2}$$

which is indeed in agreement with the QFT computation.

7. Conclusions

In this report we have shown how the CHY formalism constitutes an alternative to Feynman diagrams for the computation of tree-level scattering amplitudes. In particular, we have given a clear mathematical derivation of the Scattering Equations and their solutions. As already mentioned, the physical meaning of intermediate results, such as the individual solutions of the SE, even if present is unfortunately still unknown.

In the light of the work recently done by Dolan, Goddard, Cardona, and Kalousios to obtain explicit solutions for the Scattering Equations (discussed in sections 4 and 5), our inquiry into possible further symmetries of the SE (presented in section 3.2 and Appendix V) turned out to be superseded. Still, it was nice to obtain the expected result (no further conditions on the punctures besides the SE).
Our program in Mathematica (given in Appendix VII) allowed us to easily recover the correct matrix for the solutions to the SE for $n \leq 7$, as given by DG in [16]. Furthermore, it made the computation feasible even for systems with a higher number of particles ($n \geq 8$). Given additional time, it would be interesting to have the program compute the determinant of the matrix as well, and possibly integrate over the solutions to obtain the scattering amplitudes.

Finally, given the already present possibility of generalizing this formalism to massive, coloured and non-scalar particles as discussed in [8] and [14], future investigation could be aimed at obtaining an expression for the scattering amplitudes valid beyond tree-level. This could provide an additional tool for the computation of high precision quantities for accelerator based experiments, both within and beyond the SM. Furthermore, given the substantially different structure of this formalism compared to standard Feynman diagrams computations, it may be possible to achieve shorter computational times and push the precision limits of current predictions.

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Appendix I: The Riemann Sphere

The Riemann sphere, or extended complex plane, (Σ) is defined as the one-point compactification of the complex plane ($\mathbb{C}$) with the point at complex infinity ($\tilde{\infty}$): $\Sigma \equiv \mathbb{C} \cup \tilde{\infty}$. Complex infinity is defined as the "complex number" ($z = re^{i\phi}$) having infinite modulus ($r = \infty$) and undefined argument (ϕ). The neighbourhood of each point has to be defined as well. For all points except infinity it is trivially defined by an identity map to the complex plane. As far as the point at infinity is concerned, we send it to zero and all other points $z \neq 0$ to $1/z$.
This definition allows functions such as $f(z) = 1/z$ to be well behaved everywhere, since we can now define $1/0 = \tilde{\infty}$, with $\tilde{\infty} \in \Sigma$. Note that this is not well defined in $\mathbb{C}$ since $\tilde{\infty} \notin \mathbb{C}$. For this reason a lot of complex analysis is actually performed on the Riemann sphere rather than in the complex plane.

The topological equivalence to a sphere is most easily observed with a stereographic projection (Figure I). Consider the complex plane and a sphere centred at the origin with unit radius, such that the complex plane is the equatorial plane of the sphere. Then, for each point on the complex plane consider the straight line going though it and through the north pole of the sphere. The intersection of this line with the sphere defines an (almost) one to one correspondence between the points on the plane and those on the sphere: the origin is mapped to the south pole, points on the equator are mapped to themselves and each point $z = re^{i\phi}$, $r < 1$ ($r > 1$) goes to a point on the southern (northern) hemisphere. The only point on the sphere not having a corresponding point on the plane is the north pole itself. However, since points on the plane with $r \to \infty$ are mapped to points on the sphere closer and closer to the north pole, we say that infinity is just a point and map it to the north pole.

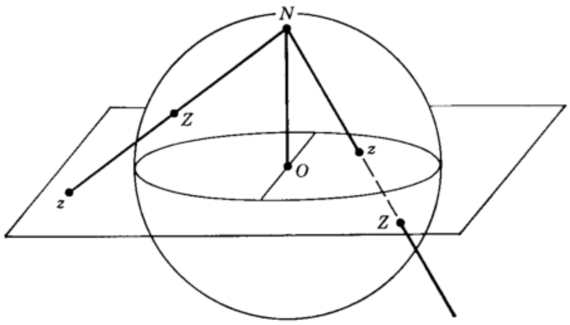

Figure I: Stereographic projection of the extendend complex plane.[2]
N is the sphere's north pole, O is the origin, z are points on the complex plane and Z are points on the sphere.

We may now introduce the Möbius transformation as an automorphism of the Riemann sphere. An automorphism is a map that takes an object to itself while preserving its structure. A Möbius transformation can be described in terms of four successive simpler maps:

- Translation $z \to \zeta = z + A$,
- Inversion $z \to \zeta = 1/z$,
- Rescaling $z \to \zeta = Bz$,
- Translation $z \to \zeta = z + C$.

Combining these and renaming the parameters we may write it in the form of equation (3.2). For further information on mappings in the complex plane see, for instance, section 5 of [19].

[2]Image courtesy of http://mathematica.stackexchange.com/questions/23793/stereographic-projection.

Appendix II: Symmetrized expression of $p^\mu(z)$

In this section we present the proof for the symmetrized expression in the punctures, σ's, and the momenta, k's, of the polynomials $p^\mu(z)$ that appear in the map from momentum space to the Riemann sphere. In particular we prove the equivalence of equations (2.4) and (2.5).

Proposition:

$$\sum_{b\in A} k_b^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_b^{n-1-m} = k_a^\mu \sum_{m=0}^{n-1} \sigma_a^m \{\sigma\}_a^{n-1-m}$$

Proof:

First of all, in analogy to eq. (2.3), define $\{\sigma\}_{b,a}^m$ *as*

$$\{\sigma\}_{b,a}^m \equiv (-1)^m \sum_{\{a_i\}\subseteq A\setminus\{a,b\}} \sigma_{a_1}\sigma_{a_2}...\sigma_{a_m}\ .$$

We can then re-express (2.3) in terms of the above:

$$\{\sigma\}_b^m = -\sigma_a \{\sigma\}_{b,a}^{m-1} + \{\sigma\}_{b,a}^m\ .$$

Note that for this expression to be valid for $\forall m \in \{1,...,n-1\}$ *we must require* $\{\sigma\}_{a,b}^{n-1} = 0$ *as well as* $\{\sigma\}_{a,b}^0 = 1$*, or give a piecewise definition for* $m = 1, n-1$.

Hence:

$$p^\mu(\sigma_a) = \sum_{b\in A} k_b^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_b^{n-1-m} = k_a^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_a^{n-1-m} + \sum_{\substack{b\in A\\ b\neq a}} k_b^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_b^{n-1-m} =$$

$$= k_a^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_a^{n-1-m} + \sum_{\substack{b\in A\\ b\neq a}} k_b^\mu \sum_{m=0}^{n-2} \sigma_a^m [-\sigma_a \{\sigma\}_{a,b}^{n-1-m-1} + \{\sigma\}_{a,b}^{n-1-m}]\ .$$

The first term is like eq. (2.4) except for the missing $m = n-1$ *term. Consider now the second term. For each* m *we have that the first term in the sum, call it* (1)*, is of order* $m+1$ *in* σ_a*, whereas the second term, call it* (2)*, is of order* m *in* σ_a*. It is then clear that:*

$$(1)|_m + (2)|_{m+1} = 0\ .$$

Furthermore, we have $(2)|_{m=0} = 0$*, as required above. Therefore, all terms cancel out except* $(1)|_{m=n-2}$ *(since there is no* $(2)|_{m=n-1}$ *to cancel it with). This leaves us with:*

$$k_a^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_a^{n-1-m} + \sum_{\substack{b\in A\\ b\neq a}} k_b^\mu \sigma_a^{n-2} [-\sigma_a \{\sigma\}_{a,b}^0] = k_a^\mu \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_a^{n-1-m} - \sigma_a^{n-1} \sum_{\substack{b\in A\\ b\neq a}} k_b^\mu\ .$$

Now, using the null conserved momenta condition (written as $\sum_{\substack{b\in A\\ b\neq a}} k_b^\mu = -k_a^\mu$*), one recovers eq. (2.4), completing the proof.*

Q.E.D.

Appendix III: The Scattering Equations

The proof here presented is for the CHY expression of the Scattering Equations. Following our discussion of section 3, we impose the condition:

$$\frac{d}{dz}p^2(z) = 0 \implies p(z)\cdot p'(z) = 0,$$

evaluated at the punctures (σ_a).

Proposition:

$$p(\sigma_a)\cdot p'(\sigma_a) = 0 \quad \iff \quad f_a(\sigma,k) = 0, \qquad \forall a \in \{1,...,n\}$$

Proof:

Using eq. (2.2) to write $p(\sigma_a)$ and differentiating eq. (2.6) with respect to z to write $p'(\sigma_a)$ we obtain: (Note that one may use (2.6) for both, but this approach makes things more concise)

$$p(\sigma_a)\cdot p'(\sigma_a) = k_a^\mu \prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\cdot\sum_{b\in A}k_b^\mu\sum_{m=0}^{n-2}m\sigma_a^{m-1}\{\sigma\}_b^{n-1-m} =$$

$$= \prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\sum_{m=1}^{n-2}m\sigma_a^{m-1}\{\sigma\}_b^{n-1-m}\ .$$

As prompted in note 1 of reference [6], we add to the above the following term:

$$\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b(n-1)\sigma_a^{n-2}\ .$$

Note that this is allowed since what we are adding is zero, explicitly:

$$\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b = k_a\cdot\sum_{b\in A}k_b = 0\ ,$$

where we have used the massless condition in the first equality and the null conserved momenta condition in the second one.

Therefore, we have:

$$\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\,[(n-1)\sigma_a^{n-2}+\sum_{m=1}^{n-2}m\sigma_a^{m-1}\{\sigma\}_b^{n-1-m}]\ .$$

Now use again the notation introduced in the previous proof:

$$\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\,[(n-1)\sigma_a^{n-2}+\sum_{m=1}^{n-2}m\sigma_a^{m-1}(-\sigma_a\{\sigma\}_{a,b}^{n-1-m-1}+\{\sigma\}_{a,b}^{n-1-m})]\ .$$

As before, call (1) *the term of order m and* (2) *the term of order $m-1$ in σ_a. Then, we have:*

$$(1)|_m+(2)|_{m+1} = -m\sigma_a^m\{\sigma\}_{a,b}^{n-1-m-1}+(m+1)\sigma_a^m\{\sigma\}_{a,b}^{n-1-m-1} = \sigma_a^m\{\sigma\}_{a,b}^{n-2-m}\ \forall m\in\{1,...,n-3\}\ ,$$

and the following from the left over terms in $m = 1$ and $m = n - 2$, respectively:

$$\sigma_a^0 \{\sigma\}_{a,b}^{n-1-1} = \{\sigma\}_{a,b}^{n-2} \quad \text{and} \quad -(n-2)\sigma_a^{n-2} \{\sigma\}_{a,b}^{n-2-(n-2)} = -(n-2)\sigma_a^{n-2} \ .$$

Combining everything:

$$\prod_{\substack{b \in A \\ b \neq a}} (\sigma_a - \sigma_b) \sum_{\substack{b \in A \\ b \neq a}} k_a \cdot k_b \, [\sigma_a^{n-2} + \sum_{m=1}^{n-3} \sigma_a^m \{\sigma\}_{a,b}^{n-2-m} + \{\sigma\}_{a,b}^{n-2}] = \prod_{\substack{b \in A \\ b \neq a}} (\sigma_a - \sigma_b) \sum_{\substack{b \in A \\ b \neq a}} k_a \cdot k_b \sum_{m=0}^{n-2} \sigma_a^m \{\sigma\}_{a,b}^{n-2-m} \ .$$

Now we have to recognize this summation as the product of $(\sigma_a - \sigma_c)$'s over all c's except $c = a, b$. We can then rewrite the above as:

$$(\prod_{\substack{b \in A \\ b \neq a}} (\sigma_a - \sigma_b))^2 \sum_{\substack{b \in A \\ b \neq a}} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0 \ .$$

Finally, we obtain the Scattering Equations, since we are assuming the punctures to be distinct, meaning the $\prod$ cannot be zero.

Q.E.D.

Appendix IV: Proof of the Möbius invariance of the SE

In the present section, we explicitly show that the Scattering Equations (eq. (3.1)) are invariant under Möbius transformations (equation (3.2), briefly discussed in Appendix I as well). For this reason we say that Möbius transformations are a symmetry of the Scattering Equations.

Proposition:

$$f_a(\sigma,k)=0 \quad\Longleftrightarrow\quad f_a(\zeta,k)=0, \qquad \forall a\in\{1,...,n\} \quad \textit{and} \quad \zeta_a=\frac{\alpha\sigma_a+\beta}{\gamma\sigma_a+\delta}$$

Proof:

Plug equation (3.2) into (3.1):

$$\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{\zeta_a-\zeta_b}=\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{\frac{\alpha\sigma_a+\beta}{\gamma\sigma_a+\delta}-\frac{\alpha\sigma_b+\beta}{\gamma\sigma_b+\delta}}=\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{\frac{(\alpha\sigma_a+\beta)(\gamma\sigma_b+\delta)-(\alpha\sigma_b+\beta)(\gamma\sigma_a+\delta)}{(\gamma\sigma_a+\delta)(\gamma\sigma_b+\delta)}}=$$

$$\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\frac{(\gamma\sigma_a+\delta)(\gamma\sigma_b+\delta)}{\cancel{\alpha\gamma\sigma_a\sigma_b}+\beta\gamma\sigma_b+\alpha\delta\sigma_a+\cancel{\beta\delta}-\cancel{\alpha\gamma\sigma_a\sigma_b}-\beta\gamma\sigma_a-\alpha\delta\sigma_b-\cancel{\beta\delta}}=\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\frac{(\gamma\sigma_a+\delta)(\gamma\sigma_b+\delta)}{(\alpha\delta-\beta\gamma)(\sigma_a-\sigma_b)}\,.$$

The first term in both numerator and denominator is independent of b and therefore can be taken out of the sum. Then, by adding and subtracting $\gamma\sigma_a$, we obtain:

$$\frac{(\gamma\sigma_a+\delta)}{(\alpha\delta-\beta\gamma)}\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\frac{(\gamma\sigma_b-\gamma\sigma_a+\gamma\sigma_a+\delta)}{(\sigma_a-\sigma_b)}=\frac{(\gamma\sigma_a+\delta)^2}{(\alpha\delta-\beta\gamma)}f_a(\sigma,k)-\frac{\gamma(\gamma\sigma_a+\delta)}{(\alpha\delta-\beta\gamma)}\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\,.$$

Now use the null conserved momenta condition $\sum_{\substack{b\in A\\ b\neq a}}k_b^\mu=-k_a^\mu$ to rewrite it as:

$$\frac{(\gamma\sigma_a+\delta)^2}{(\alpha\delta-\beta\gamma)}f_a(\sigma,k)+\frac{\gamma(\gamma\sigma_a+\delta)}{(\alpha\delta-\beta\gamma)}k_a^2\,,$$

which is indeed zero if $k_a^2=0$ (massless condition) and if $f_a(\sigma,k)=0$ (Scattering Equations).

Q.E.D.

Appendix V: Second derivative of $p(z)^2$

The algebra backing our discussion of subsection 3.2 is presented here. We compute an explicit expression for the second derivative of $p(z)^2$ evaluated at the punctures:

$$\frac{d^2}{dz^2}p(z)^2|_{z=\sigma_a} = 0 \quad \to \quad p'(\sigma_a)^2 + p(\sigma_a)\cdot p''(\sigma_a) = 0\ .$$

The first term: $p'(\sigma_a)^2$

Recalling the expression for $p'(\sigma_a)$ obtained by differentiating equation 2.6 we may write:

$$p'(\sigma_a)^2 = \sum_{b,B\in A} k_b^\mu \cdot k_B^\mu \sum_{m=1}^{n-2} m\sigma_a^{m-1}\{\sigma\}_b^{n-1-m} \sum_{M=1}^{n-2} M\sigma_a^{M-1}\{\sigma\}_B^{n-1-M} =$$

$$= \sum_{B\in A}\sum_{M=1}^{n-2} M\sigma_a^{M-1}\{\sigma\}_B^{n-1-M} \sum_{\substack{b\in A\\ b\neq B}} k_b^\mu\cdot k_B^\mu \sum_{m=1}^{n-2} m\sigma_a^{m-1}\{\sigma\}_b^{n-1-m}\ .$$

This expression looks very similar to what we have encountered in the proof of the Scattering Equations, but we must be careful because here we have $b\neq B$, not $b\neq a$. Therefore, we should distinguish between the terms $b\neq a, B$ and the term $b = a\neq B$, explicitly:

$$\sum_{B\in A}\sum_{M=1}^{n-2} M\sigma_a^{M-1}\{\sigma\}_B^{n-1-M}[\sum_{\substack{b\in A\\ b\neq a,B}} k_b^\mu\cdot k_B^\mu \sum_{m=1}^{n-2} m\sigma_a^{m-1}\{\sigma\}_b^{n-1-m} + k_a^\mu\cdot k_B^\mu\sum_{m=1}^{n-2} m\sigma_a^{m-1}\{\sigma\}_a^{n-1-m}]\ .$$

Consider now the first term in the square brackets. Similarly to what we have done in the proof for the CHY expression of the Scattering Equations, we may add

$$\sum_{B\in A}\sum_{M=1}^{n-2} M\sigma_a^{M-1}\{\sigma\}_B^{n-1-M} \sum_{\substack{b\in A\\ b\neq a,B}} k_b^\mu\cdot k_B^\mu (n-1)\sigma_a^{n-2}\ ,$$

since it is zero. Adapting the result obtained there, we get:

$$\sum_{B\in A}\sum_{M=1}^{n-2} M\sigma_a^{M-1}\{\sigma\}_B^{n-1-M} \sum_{\substack{b\in A\\ b\neq a,B}} k_b^\mu\cdot k_B^\mu \sum_{m=0}^{n-2}\sigma_a^m\{\sigma\}_{a,b}^{n-2-m}\ .$$

We can then repeat the procedure to the sum over M, which gives:

$$\sum_{\substack{b,B\in A\\ b,B\neq a}} k_b^\mu\cdot k_B^\mu \sum_{m=0}^{n-2}\sigma_a^m\{\sigma\}_{a,b}^{n-2-m}\sum_{M=0}^{n-2}\sigma_a^M\{\sigma\}_{a,B}^{n-2-M}\ ,$$

where I've also taken $B\neq a$, otherwise we are back in the other case. Finally we obtain:

$$(\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b))^2 \sum_{\substack{b,B\in A\\ b,B\neq a}} \frac{k_b\cdot k_B}{(\sigma_a-\sigma_b)(\sigma_a-\sigma_B)} = 0\ .$$

As already pointed out, the product cannot be zero for distinct puncutres. The sum is almost identical to equation 1.15 of reference [15], which was introduced by Fairlie and Roberts, and which should *vanish $\forall\sigma_a$, provided it vanishes sufficiently fast for $\sigma_a \to \infty$ and that the Scattering Equations are satisfied.*

Similarly, the terms like $b = a \neq B$ give:

$$\frac{\partial}{\partial\sigma_a}(\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b))(\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b))\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{\sigma_a-\sigma_b}=0\;,$$

which is zero provided the Scattering Equations are satisfied.

The second term: $p(\sigma_a)p''(\sigma_a)$

$$p(\sigma_a)\cdot p''(\sigma_a)=\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\sum_{m=2}^{n-2}m(m-1)\sigma_a^{m-2}\{\sigma\}_b^{n-1-m}=$$

$$=\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\sum_{m=2}^{n-2}m(m-1)\sigma_a^{m-2}[-\sigma_a\{\sigma\}_{a,b}^{n-2-m}+\{\sigma\}_{a,b}^{n-1-m}]\;.$$

Defining (1) *and* (2) *the terms of order $m-1$ and $m-2$ respectively, we may write:*

$$(1)|_m+(2)|_{m+1}=(-m^2+m)\sigma_a^{m-1}\{\sigma\}_{a,b}^{n-2-m}+(m^2+m)\sigma_a^{m-1}\{\sigma\}_{a,b}^{n-2-m}=2m\sigma_a^{m-1}\{\sigma\}_{a,b}^{n-2-m}$$
$$\forall m\in\{2,...,n-3\}\;.$$

The left over terms are:

$$(2)|_{m=2}=2\sigma_a^0\{\sigma\}_{a,b}^{n-3}=2\{\sigma\}_{a,b}^{n-3}\;,$$
$$(1)|_{m=n-2}=(n-2)(n-3)\sigma_a^{n-4}(-\sigma_a\{\sigma\}_{a,b}^0)=-(n^2-5n+6)\sigma_a^{n-3}\;.$$

All together:

$$-(n^2-5n+6)\sigma_a^{n-3}+\sum_{m=2}^{n-3}2m\sigma_a^{m-1}\{\sigma\}_{a,b}^{n-2-m}+2\{\sigma\}_{a,b}^{n-3}\;.$$

Since the first term is independent of b and is pre-multiplied by $\sum_{b\neq a}k_a\cdot k_b$ (i.e. it is zero), we can keep a factor of $2(n-2)\sigma_a^{n-3}$ only and rewrite the whole expression as follow:

$$p(\sigma_a)\cdot p''(\sigma_a)=\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}k_a\cdot k_b\,.\sum_{m=1}^{n-2}2m\sigma_a^{m-1}\{\sigma\}_{a,b}^{n-2-m}\;,$$

$$\implies p(\sigma_a)\cdot p''(\sigma_a)=2\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\frac{\partial}{\partial\sigma_a}[\prod_{\substack{b\in A\\ b\neq a}}(\sigma_a-\sigma_b)\sum_{\substack{b\in A\\ b\neq a}}\frac{k_a\cdot k_b}{\sigma_a-\sigma_b}]\;.$$

Differentiating the product gives a term proportional to the Scattering Equations, which thus vanishes. Differentiating the sum gives again something similar to equation (1.15) of reference [15], which vanishes as well without introducing new conditions.

Appendix VI: Polynomial form of the Scattering Equations

The following is an explaination of the algebra of the proof presented by Dolan and Goddard in [15] for the polynomial form of the Scattering Equations, which we discuss more qualitatively in section 4. For clarity we have divided their proof into two distinct propositions.

First proposition:

$$f_a = 0,\ \ a \in A \qquad \Longleftrightarrow \qquad g_m = 0,\ \ 2 \leq m \leq n-2$$

Proof:

First of all we have to show that g_{-1}, g_0 and g_1 vanish identically. In [15] DG explain this as a consequence of the Möbius invariance by introducing $U(\sigma,k) = \prod_{a<b}(\sigma_a - \sigma_b)^{k_a\cdot k_b}$ and taking derivatives. In truth this is quite easily showed directly from eq. (4.2) as well. Recall:

$$g_m = \sum_{a\in A} \sigma_a^{m+1} f_a = \sum_{\substack{a,b\in A\\ a\neq b}} \sigma_a^{m+1} \frac{k_a\cdot k_b}{\sigma_a - \sigma_b} \ .$$

Rewrite g_m renaming a to b and vice-versa:

$$g_m = \sum_{\substack{a,b\in A\\ a\neq b}} \sigma_b^{m+1} \frac{k_a\cdot k_b}{\sigma_b - \sigma_a} = -\sum_{\substack{a,b\in A\\ a\neq b}} \sigma_b^{m+1} \frac{k_a\cdot k_b}{\sigma_a - \sigma_b} \ ,$$

and therefore obtain:

$$g_m = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b \frac{\sigma_a^{m+1} - \sigma_b^{m+1}}{\sigma_a - \sigma_b} \ .$$

Now g_{-1} is obviously zero. The other two are given by:

$$g_0 = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b = \frac{1}{2}\sum_{a\in A} k_a \cdot \sum_{\substack{b\in A\\ b\neq a}} k_b = 0 \ ,$$

$$g_1 = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b \frac{\sigma_a^2 - \sigma_b^2}{\sigma_a - \sigma_b} = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b \frac{\cancel{(\sigma_a - \sigma_b)}(\sigma_a + \sigma_b)}{\cancel{\sigma_a - \sigma_b}} = \sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b \sigma_a =$$

$$= \sum_{a\in A} \sigma_a k_a \cdot \sum_{\substack{b\in A\\ b\neq a}} k_b = -\sum_{a\in A} \sigma_a k_a^2 = 0 \ .$$

Where relabeling, the massless and null conserved momenta conditions are used as necessary.

Going back to equation (4.2), it can be rewritten in matrix form as:

$$g_m = \Sigma_{ma} f_a \ ,$$

where Einstein's summation convention is assumed and where Σ_{ma} is defined as a $N\times N$ matrix $\Sigma_{ma} = \sigma_a^{m+1}$, $-1 \leq m \leq n-2$. Explicitly:

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_n \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 & \dots & \sigma_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{n-1} & \sigma_2^{n-1} & \sigma_3^{n-1} & \dots & \sigma_n^{n-1} \end{bmatrix} .$$

This type of matrices are named after the French mathematician Alexandre-Thophile Vandermonde and their determinant is a well known result [18]:

$$Det(\Sigma) = \prod_{1 \le a < b \le n} (\sigma_a - \sigma_b) .$$

It vanishes only for degenerate punctures, which we excluded by assumption. We can intuitively see that this is the correct form for the determinant because $\sigma_a = \sigma_b$, $a \neq b$ is the condition that makes two columns of Σ identical. A more direct proof involves the following steps:

- *Subtract to each row the previous one multiplid by σ_1*

$$|\Sigma| = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & \sigma_2 - \sigma_1 & \sigma_3 - \sigma_1 & \dots & \sigma_n - \sigma_1 \\ 0 & \sigma_2(\sigma_2 - \sigma_1) & \sigma_3(\sigma_3 - \sigma_1) & \dots & \sigma_n(\sigma_n - \sigma_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_2^{n-2}(\sigma_2 - \sigma_1) & \sigma_3^{n-2}(\sigma_3 - \sigma_1) & \dots & \sigma_n^{n-2}(\sigma_n - \sigma_1) \end{vmatrix} =$$

- *Take factors of $(\sigma_j - \sigma_1)$ out of the matrix from each column:*

$$= \begin{vmatrix} 1 & (\sigma_2 - \sigma_1)^{-1} & (\sigma_3 - \sigma_1)^{-1} & \dots & (\sigma_n - \sigma_1)^{-1} \\ 0 & 1 & 1 & \dots & 1 \\ 0 & \sigma_2 & \sigma_3 & \dots & \sigma_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_2^{n-2} & \sigma_3^{n-2} & \dots & \sigma_n^{n-2} \end{vmatrix} \prod_{j=2}^{n} (\sigma_j - \sigma_1) =$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 \\ \sigma_2 & \sigma_3 & \dots & \sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_2^{n-2} & \sigma_3^{n-2} & \dots & \sigma_n^{n-2} \end{vmatrix} \prod_{j=2}^{n} (\sigma_j - \sigma_1)$$

- *Repeat.*

The consequence of $Det(\Sigma) \neq 0$ is that the only solution to $g_m = 0$ is the trivial one $f_a = 0$, i.e. $g_m = 0$ if and only if $f_a = 0$.

Q.E.D.

Second proposition:

$$\tilde{h}_m = 0 \iff g_m = 0, \qquad 2 \le m \le n-2$$

Proof:

Start by considering the following expression:

$$\sum_{\substack{a_0,a_2,\dots,a_m\in A\\ a_i uneq.}} \sigma_{a_2}...\sigma_{a_m}\sigma_{a_0}^2 f_{a_0} = \sum_{\substack{a_0,a_2,\dots,a_m\in A\\ a_i uneq.}} \sigma_{a_2}...\sigma_{a_m}\sigma_{a_0}^2 \sum_{\substack{a_1\in A\\ a_1\neq a_0}} \frac{k_{a_0}\cdot k_{a_1}}{\sigma_{a_0}-\sigma_{a_1}} =$$

$$= \sum_{\substack{a_0,a_1,\dots,a_m\in A\\ a_i uneq.}} \frac{k_{a_0}\cdot k_{a_1}\sigma_{a_0}^2\sigma_{a_2}...\sigma_{a_m}}{\sigma_{a_0}-\sigma_{a_1}} + (m-1) \sum_{\substack{a_0,a_1,\dots,a_{m-1}\in A\\ a_i uneq.}} \frac{k_{a_0}\cdot k_{a_1}\sigma_{a_0}^2\sigma_{a_1}...\sigma_{a_{m-1}}}{\sigma_{a_0}-\sigma_{a_1}} =$$

The first term is the case where a_1 is different from all other a_i with $2 \le i \le m$, whereas the second term is the case where a_1 is equal to one of the $(m-1)$ a's, and hence the multiplicative pre-factor.

$$= \sum_{\substack{a_0,a_1,\dots,a_m\in A\\ a_i uneq.}} k_{a_0}\cdot k_{a_1}\sigma_{a_0}\sigma_{a_2}...\sigma_{a_m} + \frac{m-1}{2} \sum_{\substack{a_0,a_1,\dots,a_{m-1}\in A\\ a_i uneq.}} k_{a_0}\cdot k_{a_1}\sigma_{a_0}\sigma_{a_1}...\sigma_{a_{m-1}} =$$

This is obtained by performing polynomial division, esplicitelly:

$$\frac{\sigma_{a_0}^2}{\sigma_{a_0}-\sigma_{a_1}} = \sigma_{a_0}+\sigma_{a_1}+\frac{\sigma_{a_1}^2}{\sigma_{a_0}-\sigma_{a_1}} \;=\!=\; relabeling \Longrightarrow \quad \frac{\sigma_{a_0}^2}{\sigma_{a_0}-\sigma_{a_1}} = \sigma_{a_0}\ ,$$

which gives the first term, and:

$$\frac{\sigma_{a_0}^2\sigma_{a_1}}{\sigma_{a_0}-\sigma_{a_1}} = \sigma_{a_0}\sigma_{a_1}+\frac{\sigma_{a_0}\sigma_{a_1}^2}{\sigma_{a_0}-\sigma_{a_1}} \;=\!=\; relabeling \Longrightarrow \quad \frac{\sigma_{a_0}^2\sigma_{a_1}}{\sigma_{a_0}-\sigma_{a_1}} = \frac{1}{2}\sigma_{a_0}\sigma_{a_1}\ ,$$

which gives the second one.
Now perform the sum over a_1 in the first term. By writing it as

$$\sum_{\substack{a_1\in A\\ a_1\neq a_i\\ i=0,2,3,\dots,n}} k_{a_1} = -k_{a_0}\underbrace{-k_{a_2}-k_{a_3}\cdots-k_{a_n}}_{(n-1)terms}\ ,$$

and relabeling, we obtain: ($k_{a_0}^2 = 0$ is also used)

$$= -\frac{m-1}{2} \sum_{\substack{a_0,a_1,\dots,a_{m-1}\in A\\ a_i uneq.}} k_{a_0}\cdot k_{a_1}\sigma_{a_0}\sigma_{a_1}...\sigma_{a_{m-1}} = -\frac{1}{m} \sum_{\substack{a_1,a_2,\dots,a_m\in A\\ a_i uneq.}} \sum_{\substack{i,j=1\\ i<j}}^{m} k_{a_i}\cdot k_{a_j}\sigma_{a_1}\sigma_{a_2}...\sigma_{a_m} =$$

where the second equality is obtained by simply relabeling and counting: the $\sum_{i,j}$ gives $\frac{m(m-1)}{2}$ terms like the previous ones.
Now introduce the following short-hand notation: $k_{a_1a_2a_3...a_m} = k_{a_1}+k_{a_2}+k_{a_3}+\cdots+k_{a_m}$. This allows to rewrite the above as:

$$= -\frac{1}{2m} \sum_{\substack{a_1,a_2,\dots,a_m\in A\\ a_i uneq.}} k_{a_1a_2a_3...a_m}^2\sigma_{a_1}\sigma_{a_2}...\sigma_{a_m} =$$

and finally:

$$= -\frac{(m-1)!}{2} \sum_{\substack{S\subset A \\ |S|=m}} k_S^2 \sigma_S \ .$$

Since the two are identical, up to counting. The $\sum_{\substack{a_1,a_2,\ldots,a_m\in A \\ a_i uneq.}}$ *has* $n(n-1)(n-2)\ldots(n-m+1) = \frac{n!}{(n-m)!}$ *terms, whereas the* $\sum_{\substack{S\subset A \\ |S|=m}}$ *has* $C_m^n = \frac{n!}{m!(n-m)!}$ *terms.*

Summing up:

$$\sum_{\substack{a_0,a_2,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_2}...\sigma_{a_m}\sigma_{a_0}^2 f_{a_0} = -\frac{(m-1)!}{2}\tilde{h}_m \ . \tag{II.2.1}$$

Now relating the above sum to g_m *will give a relation between* $\tilde{h}_m$ *and* g_m*.*

$$\sum_{\substack{a_0,a_2,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_2}...\sigma_{a_m}\sigma_{a_0}^2 f_{a_0} = \sum_{\substack{a_2,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_2}...\sigma_{a_m} \sum_{a_0\in A} \sigma_{a_0}^2 f_{a_0} - (m-1) \sum_{\substack{a_0,a_3,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_3}...\sigma_{a_m}\sigma_{a_0}^2 f_{a_0} \ .$$

DG reasoning here is that the sum over a_0 *not equal to the other* a*'s is equal to the sum over all* a_0 *minus the sum of* a_0 *equal to the other* a*'s (of which there are* $m-1$*). This procedure can be repeated:*

$$= \sum_{\substack{a_2,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_2}...\sigma_{a_m} \sum_{a_0\in A} \sigma_{a_0}^2 f_{a_0} - (m-1)\Big[\sum_{\substack{a_3,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_3}...\sigma_{a_m} \sum_{a_0\in A} \sigma_{a_0}^2 f_{a_0} -$$

$$(m-2) \sum_{\substack{a_0,a_4,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_4}...\sigma_{a_m}\sigma_{a_0}^2 f_{a_0}\Big] = \cdots =$$

In general:

$$= \sum_{r=2}^{m+1} \frac{(-1)^r (m-1)!}{(m-r+1)!} \sum_{\substack{a_r,\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_r}...\sigma_{a_m} \sum_{a_0\in A} \sigma_{a_0}^r f_{a_0} =$$

We want to relate this to equation (4.2). Let $r \to r+1$ *and recall that* g_1 *is identically zero:*

$$= \sum_{r=2}^{m} \frac{(-1)^{r+1}(m-1)!}{(m-r)!} \sum_{\substack{a_{r+1},\ldots,a_m\in A \\ a_i uneq.}} \sigma_{a_{r+1}}...\sigma_{a_m} g_r =$$

$$= (m-1)! \sum_{r=2}^{m} (-1)^{r+1} \Sigma_{m-r}^A g_r \ , \tag{II.2.2}$$

where Σ_r^A *is defined as necessary:*

$$\Sigma_r^A = \sum_{\substack{S\subset A \\ |S|=r}} \sigma_S = \frac{1}{r!} \sum_{\substack{a_1,\ldots,a_r\in A \\ a_i uneq.}} \sigma_{a_1}...\sigma_{a_r} \ .$$

Now we can straightforewardly combine (II.2.1) with (II.2.2) to yield:

$$\tilde{h}_m = \sum_{r=2}^{m} (-1)^r g_r \Sigma^A_{m-r}, \quad 2 \leq m \leq n-2 \,. \tag{II.2.3}$$

In matrix form:

$$\begin{bmatrix} \tilde{h}_2 \\ \tilde{h}_3 \\ \tilde{h}_4 \\ \vdots \\ \tilde{h}_{n-2} \end{bmatrix} = \begin{bmatrix} 2\Sigma^A_0 & 0 & 0 & \dots & 0 \\ 2\Sigma^A_1 & -2\Sigma^A_0 & 0 & \dots & 0 \\ 2\Sigma^A_2 & -2\Sigma^A_1 & 2\Sigma^A_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\Sigma^A_{n-4} & -2\Sigma^A_{n-5} & 2\Sigma^A_{n-6} & \dots & 2(-1)^{n-2}\Sigma^A_0 \end{bmatrix} \begin{bmatrix} g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_{n-2} \end{bmatrix} .$$

The determinant - as for any lower triangolar matrix - is simply the product of the elements in the main diagonal. Since $\Sigma^A_0 = 1$ this determinant is non-zero and therefore the $\tilde{h}$'s are equivalent to the g's (and subsequently to the f_a's).

Q.E.D.

Appendix VII: Program to solve the Scattering Equations

The aim of this section of the appendix is to present work done in order to obtain solutions to the Scattering Equations and hopefully convey to the reader the scaling of the complexity of the problem as a function of the number of particles n.

The first version of the program followed the general n elimination theory algorithm very closely. It explicitly construed the vector V of the $(n-3)!$ variables we wish to eliminate and, by taking partial derivatives, obtained the $(n-3)! \times (n-3)!$ matrix whose determinant is the equation we are looking for. We later modified it to exploit both the recursive relation and the cyclic permutation structure of the matrix. The difference turned out to be huge, as shown in the following table.

# of particles n	size of the matrix	timing 1^{st} version	timing 2^{nd} version
6	6×6	$< 0.1s$	$< 0.1s$
7	24×24	$0.2s$	$< 0.1s$
8	120×120	$13s$	$0.7s$
9	720×720	$38m$	$4s$
10	5040×5040	$> 2h$	$2m40s$

Table 1: Comparison of the time required by the two versions of the Mathematica program to obtain the matrix whose determinant gives the solutions of the Scattering Equations.

The program itself and sample outputs for the n=7 and n=8 case are shown in the next pages. The outputs shown consist of, in order: the set of cyclic permutation for the rows of the matrix; the matrix itself in the abbreviated notation used in section 5[3]; the Scattering Equations; and the first set of rows of the matrix written explicitly in terms of the Scattering Equations and their derivatives.

[3]For the n=8 case only the first half of the matrix is shown. Showing the full output would make it too small to read.

The program:

```
(* CHOOSE HOW MANY PARTICLES n≥4 *)
n = 7;

(* RECURSIVE RELATION FOR THE MATRIX *)
M[i_] := If[i == 4, 1,
  ArrayFlatten[Table[If[r == c || r == (c - 1), If[r == c, M[i - 1], σ_(i-3) M[i - 1]],
     Table[0, {l, (i - 5) !}, {m, (i - 4) !}]] , {r, (i - 4)}, {c, (i - 3)}], 2]]
Ma = M[n];
MatrixForm[Ma]

(* CALCULATE THE PERMUTATION ASSOCIATED WITH
 EACH SET OF ROWS BY LOOKING AT THE POSITION OF "1" *)
P = {0};
For[i = 2, i ≤ (n - 4) !, i ++,
 For[j = 1, j ≤ (n - 3) !, j ++,
   If[Extract[Extract[Ma, {i}], {j}] == 1, P = Append[P, (j - 1)];
    Break[];]];
]
Print[P]

(* TAKES THE FIRST ROW OF THE MATRIX, WHICH IS ALL WE NEED *)
R1 = Extract[Ma, {1}];

(* WRITE DOWN EXPLICITLY THE SCATTERING EQUATIONS *)
(* SINCE σ_1→∞ AND σ_n→0 DEFINE THE SET "A" AS FOLLOWS: *)
A = {2, 3};
For[nn = 4 , nn < n, nn ++, A = Union[A, {nn}]]
(* WRITE h_m WITH SET NOTATION *)
For[m = 1, m <= (n - 3) , m ++,
 h_m = 0;
  (* this is a sum over all possible subsets of A *)
 For[j = 1, j ≤ Binomial[(n - 2) , m] , j ++,
      h_m = h_m + k_Union[{1},Extract[Subsets[A,{m,m}],{j}]] ^2 σ_Extract[Subsets[A,{m,m}],{j}] ]]

(* CLEAR THE NOTATION h_m ARE WRITTEN IN *)
f1[σ_S_] :=
 Product[σ_Extract[Extract[Subsets[S,{1,1}],{i}],{1}] , {i, 1, Length[Subsets[S, {1, 1}]]}]
f0[k_S_^2 σ_T_] := k_S^2 f1[σ_T]
For[m = 1, m ≤ (n - 3) , m ++,
 h_m = Map[f0, h_m];
 Print[ h_m];
]

(* COMPUTES THE FIRST n-3 ROWS OF THE FINAL MATRIX
  IN TERMS OF THE DERIVATIVES OF THE SCATTERING EQUATIONS *)
Clear[Mb]
Mb = {0};
For[j = 1, j ≤ (n - 3) , j ++,
```

```
 For[i = 1, i ≤ (n - 3) !, i ++,
  If[Extract[R1, {i}] == 0, Mb = Append[Mb, 0], Mb = Append[Mb, h_j],
   t = h_j;
   For[l = 1, l ≤ (n - 3), l ++,
    If[D[Extract[R1, {i}], σ_l] ≠ 0, t = D[t, σ_l],  , t = D[t, σ_l]]
   ];
   Mb = Append[Mb, t]]
 ]
]
Mb = Drop[Mb, 1];
Mb = Partition[Mb, (n - 3) !];
MatrixForm[Mb]

(* SIMPLIFIES THE EXPRESSION BY SETTING ALL
 PUNCTURES EXCEPT THE TWO WE ARE INTERESTED IN TO ZERO *)
Mb = Flatten[Mb];
For[j = 1, j ≤ (n - 3), j ++,
 For[i = 1, i ≤ (n - 3) !, i ++,
  If[Extract[Mb, {(j - 1) * (n - 3) ! + i}] ≠ 0, , ,
   For[l = 1, l ≤ (n - 3), l ++,
    Mb = ReplacePart[Mb,
      ((j - 1) * (n - 3) ! + i) → Extract[Mb, {(j - 1) * (n - 3) ! + i}] /. σ_l → 0]
   ]
  ]
 ]
]
Mb = Partition[Mb, (n - 3) !];
MatrixForm[Mb]

(* RECONSTRUCTS THE FULL MATRIX THANKS
 TO THE PREVIOUSLY COMPUTED PERMUTATIONS *)
Mc = Mb;
For[j = 2, j ≤ (n - 4) !, j ++,
 For[i = 1, i ≤ (n - 3), i ++,
  Mc = Append[Mc, RotateRight[Extract[Mb, {i}], Extract[P, {j}]]]
 ]
]
MatrixForm[Mc]
```

Output for n=7 (a) and partial output for n=8 (b):

(a)

{0, 2, 6, 8, 12, 14}

$$\begin{pmatrix} H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 & H^{\sigma_4} & H^{\sigma_2\sigma_4} & H^{\sigma_3\sigma_4} & H^{\sigma_2\sigma_3\sigma_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 & H^{\sigma_4} & H^{\sigma_2\sigma_4} & H^{\sigma_3\sigma_4} & H^{\sigma_2\sigma_3\sigma_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 & H^{\sigma_4} & H^{\sigma_2\sigma_4} & H^{\sigma_3\sigma_4} & H^{\sigma_2\sigma_3\sigma_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 & H^{\sigma_4} & H^{\sigma_2\sigma_4} & H^{\sigma_3\sigma_4} & H^{\sigma_2\sigma_3\sigma_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 & H^{\sigma_4} & H^{\sigma_2\sigma_4} & H^{\sigma_3\sigma_4} & H^{\sigma_2\sigma_3\sigma_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & H^{\sigma_2} & H^{\sigma_3} & H^{\sigma_2\sigma_3} & 0 & 0 & H^{\sigma_4} & H^{\sigma_2\sigma_4} & H^{\sigma_3\sigma_4} & H^{\sigma_2\sigma_3\sigma_4} \end{pmatrix}$$

$k^2_{\{1,2\}}\,\sigma_2 + k^2_{\{1,3\}}\,\sigma_3 + k^2_{\{1,4\}}\,\sigma_4 + k^2_{\{1,5\}}\,\sigma_5 + k^2_{\{1,6\}}\,\sigma_6$

$k^2_{\{1,2,3\}}\,\sigma_2\sigma_3 + k^2_{\{1,2,4\}}\,\sigma_2\sigma_4 + k^2_{\{1,3,4\}}\,\sigma_3\sigma_4 + k^2_{\{1,2,5\}}\,\sigma_2\sigma_5 + k^2_{\{1,3,5\}}\,\sigma_3\sigma_5 + k^2_{\{1,4,5\}}\,\sigma_4\sigma_5 + k^2_{\{1,2,6\}}\,\sigma_2\sigma_6 + k^2_{\{1,3,6\}}\,\sigma_3\sigma_6 + k^2_{\{1,4,6\}}\,\sigma_4\sigma_6 + k^2_{\{1,5,6\}}\,\sigma_5\sigma_6$

$k^2_{\{1,2,3,4\}}\,\sigma_2\sigma_3\sigma_4 + k^2_{\{1,2,3,5\}}\,\sigma_2\sigma_3\sigma_5 + k^2_{\{1,2,4,5\}}\,\sigma_2\sigma_4\sigma_5 + k^2_{\{1,3,4,5\}}\,\sigma_3\sigma_4\sigma_5 + k^2_{\{1,2,3,6\}}\,\sigma_2\sigma_3\sigma_6 + k^2_{\{1,2,4,6\}}\,\sigma_2\sigma_4\sigma_6 + k^2_{\{1,3,4,6\}}\,\sigma_3\sigma_4\sigma_6 + k^2_{\{1,2,5,6\}}\,\sigma_2\sigma_5\sigma_6 + k^2_{\{1,3,5,6\}}\,\sigma_3\sigma_5\sigma_6 + k^2_{\{1,4,5,6\}}\,\sigma_4\sigma_5\sigma_6$

$k^2_{\{1,2,3,4,5\}}\,\sigma_2\sigma_3\sigma_4\sigma_5 + k^2_{\{1,2,3,4,6\}}\,\sigma_2\sigma_3\sigma_4\sigma_6 + k^2_{\{1,2,3,5,6\}}\,\sigma_2\sigma_3\sigma_5\sigma_6 + k^2_{\{1,2,4,5,6\}}\,\sigma_2\sigma_4\sigma_5\sigma_6 + k^2_{\{1,3,4,5,6\}}\,\sigma_3\sigma_4\sigma_5\sigma_6$

$$\begin{pmatrix} k^2_{\{1,5\}}\sigma_5 + k^2_{\{1,6\}}\sigma_6 & k^2_{\{1,2\}} & k^2_{\{1,3\}} & 0 & 0 & 0 & k^2_{\{1,4\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k^2_{\{1,5,6\}}\sigma_5\sigma_6 & k^2_{\{1,2,5\}}\sigma_5 + k^2_{\{1,2,6\}}\sigma_6 & k^2_{\{1,3,5\}}\sigma_5 + k^2_{\{1,3,6\}}\sigma_6 & k^2_{\{1,2,3\}} & 0 & 0 & k^2_{\{1,4,5\}}\sigma_5 + k^2_{\{1,4,6\}}\sigma_6 & k^2_{\{1,2,4\}} & k^2_{\{1,3,4\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k^2_{\{1,2,5,6\}}\sigma_5\sigma_6 & k^2_{\{1,3,5,6\}}\sigma_5\sigma_6 & k^2_{\{1,2,3,5\}}\sigma_5 + k^2_{\{1,2,3,6\}}\sigma_6 & 0 & 0 & k^2_{\{1,4,5,6\}}\sigma_5\sigma_6 & k^2_{\{1,2,4,5\}}\sigma_5 + k^2_{\{1,2,4,6\}}\sigma_6 & k^2_{\{1,3,4,5\}}\sigma_5 + k^2_{\{1,3,4,6\}}\sigma_6 & k^2_{\{1,2,3,4\}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k^2_{\{1,2,3,5,6\}}\sigma_5\sigma_6 & 0 & 0 & 0 & k^2_{\{1,2,4,5,6\}}\sigma_5\sigma_6 & k^2_{\{1,3,4,5,6\}}\sigma_5\sigma_6 & k^2_{\{1,2,3,4,5\}}\sigma_5 + k^2_{\{1,2,3,4,6\}}\sigma_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b)